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MULTIFRACTAL SCALING FOR RISKY ASSET MODELLING

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Abstract. This paper reviews a class of multifractal models obtained via products of exponential Ornstein-Uhlenbeck processes driven by Lévy motion. Given a self-decomposable distribution, conditions for constructiong multifractal scenarios and general formulas for their Renyi functions are provided. Together with several examples, a model with multifractal activity time is discussed ad application to exchange data is presented.

Key words: Multifractal process, Ornstein-Uhlenbeck process, self-decomposable distribution, subordinator, Renyi function, Exchange rates.

Mathematics Subject Classification: 62M10, 60G57, 60G17.

1 Introduction

Since Mandelbrot (in particular 1977, 1982, and 1997) developed and popularized the concept of fractals and multifractals, and advocated their use in the explanation of observed features of time series arising in natural sciences, there has been ongoing interest by researchers in a variety of disciplines in widening their application.

The paradigm option pricing formula of Black and Scholes (discovered in 1973), and the ensuing arbitrage-free methodology which it has engendered, has occupied a central place in asset-liability management, theory and practice. While the underlying geometric Brownian motion (GBM) model of their formula surely captured the essence of option pricing, three decades of econometric investigation have shown that the model departs from the realities of risky asset price and risky asset returns (increments in log-price) data in quite a number of important ways. The empirical characteristics of historical returns data such as; little or no correlation but evidence of some dependence, and a marginal distribution with higher peaks and heavier tails than Gaussian are universally accepted (see Granger (2005)) and has led to invalidation.

It has been shown, see for example Schmitt et al (1999), Calvet and Fisher (2002), how, through the use of fractals (in particular, multifractals) in a finance setting, how one can too remedy some of the empirically established and occasionally puzzling shortcomings of the original option pricing model. In this paper we are going to discuss a class of multifractal models originally introduced by Anh et al (2008) and show they provide a useful and flexible family of models for applications.

We will give a short description of the main features of fractals and multifractals in the next section. Section 3 will introduce an alternative model to the classical GBM model, and show that we can incorporate multiscaling. In section 4 we equip the model with a multifractal process construction based on the products of geometric Ornstein-Uhlenbeck (OU) processes. We then consider five cases of infinitely divisible distributions for the background driving process including their Rényi function and dependence structure. Section 5 shows empirical evidence that multifractality exists for real financial data through the nonlinear nature of the scaling function. We then validate this new approach by testing the fit of the model (Rényi function) to the data (scaling function).

2 Multifractals

There are two main models for fractals that occur in nature. Generally speaking, fractals are either statistically self-similar or they are multifractals.

Multifractals were introduced in Mandelbrot (1972) as measures to model turbulence. The concept was extended in Mandelbrot et al. (1997) to stochastic processes as a generalisation of self-similar stochastic processes. The definition of a multifractal is motivated by that of a stochastic process X_t which satisfies a relationship of the form

$$\{X(ct)\} \stackrel{d}{=} \{M(c)X(t)\}, \ t \geq 0 \tag{2.1}$$

for positive $0 < c < 1$ where M is a random variable independent of X and equality is in finite-dimensional distributions.

In the special case $M(c) = c^H$, the multifractal reduces to a self-similar fractal where the parameter $0 < H < 1$ is known as the Hurst parameter named after the British engineer Harold Hurst (whose work on Nile river data played an important role in the development of self-similar processes). For a more detailed review of self-similar processes see Embrechts and Maejima (2002).

It is assumed further that

$$M(ab) = M_1(a)M_2(b), \ a, b > 0,$$

where M_1 and M_2 are independent random variables with common distribution M.

The actual definition of a multifractal process, as given in Mandelbrot et al. (1997), is defined in terms of the moments of the process and includes processes satisfying (2.1):

A stochastic process $X(t)$ is multifractal if it has stationary increments and there exist functions $c(q)$ and $\tau(q)$ and positive constants Q and T such that $\forall q \epsilon Q = [q_-, q_+], \forall t \epsilon [0, T]$,

$$E(|X(t)|^q) \stackrel{d}{=} c(q) t^{\tau(q)+1}, \tag{2.2}$$

where $\tau(q)$ and $c(q)$ are both deterministic functions of q. $\tau(q)$ is called the scaling function and takes into account the influence of the time t on the moments q, and $c(q)$ is called the prefactor.

While this definition is the standard definition of a multifractal process most processes studied as multifractals only obey it for particular values of t or sometimes for asymptotically small t. The condition of stationary increments is also quite often relaxed.

Conversely, Taqqu et al (1997) tests the scaling properties of the increments of $X(t)$ instead of the process itself. If this method is used then the subtraction of the mean $E(X(t+1) - X(t))$ to $X(t+1) - X(t)$ may be required to ensure a fair investigation, because such a stationary process cannot be self-similar or even asymptotically self-similar if it has non-zero mean. For our case, we find that $E(X(t+1) - X(t)) = 0$ for each of our data sets.
It follows from (2.2) that

$$\log E(|X(t)|^q) = \log c(q) + (\tau(q) + 1) \log t$$

and so $X(t)$ is multifractal if for each $q \in Q$, $\log E|X(t)|^q$ scales linearly with $\log t$ and the slope is $\tau(q) + 1$. This will become the primary test used to determine if a process is multifractal.

To explain the notion of the scaling function $\tau(q)$, consider the particular case of the fractional Brownian motion- a self-similar process. A fractional Brownian motion, with a Hurst exponent H, satisfies

$$X(t) \stackrel{d}{=} t^H X(1),$$

which implies that

$$E(|X(t)|^q) \stackrel{d}{=} t^{Hq} E(|X(1)|^q).$$

Here we obtain the prefactor

$$c(q) = E(|X(1)|^q),$$

and the scaling function

$$\tau(q) = Hq - 1.$$

So the scaling function is linear if the process is self-similar. Alternatively, the process is multifractal if it has the multiscaling properties that imply nonlinearity of the scaling function.

Mandelbrot et al. (1997) showed that the scaling function is concave for all multifractals with the following argument. Let ω_1, ω_2 be positive weights with $\omega_1 + \omega_2 = 1$ and let $0 \leq q_1, q_2 \leq q_+$ and $q = q_1\omega_1 + q_2\omega_2$. Then by Hölder inequality

$$E|X(t)|^q \leq (E|X(t)|^{q_1})^{\omega_1}(E|X(t)|^{q_2})^{\omega_2}$$

and so

$$\log c(q) + \tau(q) \log t \leq (\omega_1\tau(q_1) + \omega_2\tau(q_2)) \log t + (\omega_1 \log c(q_1) + \omega_2 \log c(q_2))$$

Letting t go to zero we have $\tau(q) \geq \omega_1\tau(q_1) + \omega_2\tau(q_2)$ so τ is concave. If $T = \infty$ we can let t go to ∞ and we get the reverse inequality $\tau(q) \leq \omega_1\tau(q_1) + \omega_2\tau(q_2)$. It follows that $T = \infty$ implies that τ is linear and so $X(t)$ is self-similar.

An important associated concept is the multifractal spectrum. It is the Legendre transform of the scaling function $\tau(q)$ and is given by

$$f(\alpha) = \inf_q [q\alpha - \tau(q)].$$

where it is defined. For self-similar processes it is only defined at H with f(H)=1. The multifractal spectrum plays an important role in multifractal measures where it represents the fractal dimensions of sets where the measure has certain limiting intensities. The analogous definition for multifractal processes is the dimension of sets with local Hölder exponent α (see Calvet et al. (1997) for details). However, for multifractal processes the multifractal spectrum is only used as a tool for fitting the model to data.

The motivating example of a multifractal process is the cascade. They were first introduced as measures in Mandelbrot (1974) and can be defined on the interval $[0, 1]$ as follows. Define a sequence of random measures μ_n by

$$\mu_n(dt) = \prod_{i=1}^{n} M_{\eta_1,\eta_2,\ldots,\eta_i}(dt)$$

where t has expansion $t = 0.\eta_1\eta_2\ldots$ in base b and the $M_{\eta_1,\eta_2,\ldots,\eta_i}$ are a collection of positive iid random variables with distribution M where $EM = 1$. Kahane and Peyrière (1976) showed that the almost sure vague limit of μ_n exists, denoted as μ. The stochastic process $X(t)$ is defined as $X(t) = \mu([0, t])$. It is easy to check that (2.2) holds when $t = b^{-n}$. Of course $X(t)$ does not fully satisfy the definition of a multifractal as equation (2.2) does not hold except when t is of the form b^{-n} and $X(t)$ does not even have stationary increments. Even though cascades do not satisfy the formal definition they remain the prototype model for multifractal processes.

Multifractals overcome an important limitation of self-similar stochastic processes which is they can be positive and still have finite mean as in the case of cascades. When $X(t)$ is positive and $EX(1) < \infty$ equation (2.2) implies that $\tau(1) = 0$.

3 The Risky Asset Model

This section investigates the scaling behaviour of two models for the price of risky assets like equities, indices, foreign exchange, commodities etc. For the remainder of the paper, let $P(t)$ be the price of a risky asset and time t, and the corresponding log-price process be given by $\{Y(t) = \log P(t) - \log P(0), 0 \leq t \leq T\}$.

Under the GBM model (Black and Scholes (1973)), with zero drift term,

$$Y(t) = \sigma B(t), \ t \geq 0,$$

where $\sigma > 0$ is a fixed constant and $\{B(t), t \geq 0\}$ is a standard Brownian motion which is self-similar with $H = \frac{1}{2}$. By the scaling property of standard Brownian motion

$$\begin{aligned} E(|Y(t)|^q) &\stackrel{d}{=} t^{\frac{q}{2}} E(|\sigma B(1)|^q) \\ &= t^{\frac{q}{2}} (\sqrt{2\sigma^2})^q \frac{\Gamma(\frac{1+q}{2})}{\sqrt{\pi}}, q \geq 0. \end{aligned}$$

Note here that the scaling function $\tau_Y(q) = \frac{q}{2} - 1, \ q \geq 0$ is linear.

A subordinator model based on fractal activity time was first introduced in Heyde (1999) with the primary aim to encompass the empirically found characteristics of financial data. Under this model, again with zero drift,

$$Y(t) = \sigma B(A(t)), \ t \geq 0 \tag{3.1}$$

where $\sigma > 0$ is a fixed constant, and $\{B(t), t \geq 0\}$ is a standard (or could be extended to fractional, see various authors including Mandelbrot et al. (1997) and Elliott and Van Der Hoek (2003)) Brownian motion and $\{A(t)\}$ is a random process that is independent of $\{B(t)\}$. Here $\{A(t)\}$ is a nondecreasing stochastic process with stationary (but not necessarily independent) increments which models the underlying market activity time rather than "clock time". We assume that $EA_t = t$ and so as t goes to infinity it follows from the ergodic theorem that $\frac{1}{t}A_t \to 1$ almost surely. For a discussion of subordinator models see Rachev and Mittnik (2000).

From Heyde (1999) and Heyde and Liu (2001), sufficient empirical evidence exists to suggest that $\{A(t) - t\}$ is asymptotically self-similar. By further investigation, it has been proved in Heyde and Leonenko (2005) that $\{A(t)\}$ itself cannot be self-similar. Clearly, if $\{A(t), t \geq 0\}$ is a multifractal process with scaling function $\tau_A(q)$, by (2.2),

$$\begin{aligned} E(|Y(t)|^q) &\stackrel{d}{=} EA(t)^{\frac{q}{2}} E(|\sigma B(1)|^q) \\ &\stackrel{d}{=} c_A(\frac{q}{2}) t^{\tau_A(\frac{q}{2})+1} E(|\sigma B(1)|^q). \\ &= c_A(\frac{q}{2}) t^{\tau_A(\frac{q}{2})+1} (\sqrt{2\sigma^2})^q \frac{\Gamma(\frac{1+q}{2})}{\sqrt{\pi}}, \ q \geq 0. \end{aligned} \tag{3.2}$$

So the scaling function is given by

$$\tau_Y(q) = \tau_A(\frac{q}{2}), \ q \geq 0,$$

and this leads us directly to the following empirical test:

- if the scaling function $\tau(q)$ is linear then the process is self-similar.
- if the scaling function $\tau(q)$ is non-linear then the process is multifractal (always concave).

4 Constructing the Multifractal Process

Models with multifractal scaling have been used in many applications in hydrodynamic turbulence, genomics, computer network traffic, etc. (see Kolmogorov (1941, 1962), Gupta and Waymire (1993), Novikov (1994), Frisch (1995), Anh et al (2001)). The application to finance was first investigated by Mandelbrot et al (1997), where it is established that most multifractal models are not designed to cover important features of financial data, such as a tractable dependence structure.

To surmount these problems, Anh et al (2008) considered multifractal products of stochastic processes as defined in Kahane (1985, 1987) and Mannersalo et al (2002). These multifractals are based on products of geometric Ornstein-Uhlenbeck processes driven by Lévy motion were constructed, and several cases of infinitely divisible distributions for the background driving Lévy process are studied. The behaviour of the q-th order moments and Rényi functions were found to be nonlinear, hence displaying the multifractality as required. We will replicate this methodology and look to integrate this construction into the model (3.1).

4.1 Multifractal products of stochastic processes

We begin by recapturing some basic results on multifractal products of stochastic processes as developed in Kahane (1985, 1987) and Mannersalo et al (2002). The following conditions hold:

C1 Let $\Lambda(t),\ \ t \in \mathbb{R}_+ = [0, \infty)$, be a measurable, separable, strictly stationary, positive stochastic process with $\mathrm{E}\Lambda(t) = 1$.

We call this process the mother process and consider the following setting:

C2 Let $\Lambda^{(i)},\ i = 0, 1, ...$ be independent copies of the mother process Λ, and $\Lambda_b^{(i)}$ be the rescaled version of $\Lambda^{(i)}$

$$\Lambda_b^{(i)}(t) \stackrel{d}{=} \Lambda^{(i)}(tb^i), \quad t \in \mathbb{R}_+, \quad i = 0, 1, 2, \ldots,$$

where the scaling parameter $b > 1$.

C3 For $t \in \mathbb{R}_+$, let $\Lambda(t) = \exp\{X(t)\}$, where $X\ (t)$ is a stationary process with $\mathrm{E}X^2(t) < \infty$.

We denote by $\theta \in \Theta \subseteq \mathbb{R}^p, p \geqslant 1$ the parameter vector of the distribution of the process $X(t)$ and assume that there exist a marginal probability density function $p_\theta(x)$ and a bivariate probability density function $p_\theta(x_1, x_2; t_1 - t_2)$ such that the moment generating function

$$M(\zeta) = \mathrm{E}\exp\{\zeta X(t)\}$$

and the bivariate moment generating function

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}$$

exist.

The conditions **C1-C3** yield

$$\begin{aligned} \mathrm{E}\Lambda_b^{(i)}(t) &= M(1) = 1; \\ \mathrm{Var}\Lambda_b^{(i)}(t) &= M(2) - 1 = \sigma_\Lambda^2 < \infty; \end{aligned}$$

$$\mathrm{Cov}(\Lambda_b^{(i)}(t_1), \Lambda_b^{(i)}(t_2)) = M(1, 1; (t_1 - t_2)b^i) - 1, \;\; b > 1.$$

We define the finite product processes

$$\Lambda_n(t) = \prod_{i=0}^{n} \Lambda_b^{(i)}(t) = \exp\left\{\sum_{i=0}^{n} X(tb^i)\right\}, \tag{4.1}$$

and the cumulative processes

$$A_n(t) = \int_0^t \Lambda_n(s)ds, \quad n = 0, 1, 2, \ldots \tag{4.2}$$

We also consider the corresponding positive random measures defined on Borel sets B of $\mathbb{R}_+$

$$\mu_n(B) = \int_B \Lambda_n(s)ds, \quad n = 0, 1, 2, \ldots \tag{4.3}$$

Kahane (1987) proved that the sequence of random measures μ_n converges weakly almost surely to a random measure μ. Moreover, given a finite or countable family of Borel sets B_j on $\mathbb{R}_+$, it holds that $\lim_{n\to\infty}\mu_n(B_j) = \mu(B_j)$ for all j with probability one. The almost sure convergence of $A_n(t)$ in countably many points of $\mathbb{R}_+$ can be extended to all points in $\mathbb{R}_+$ if the limit process $A(t)$ is almost surely continuous. In this case, $\lim_{n\to\infty} A_n(t) = A(t)$ with probability one for all $t \in \mathbb{R}_+$. As noted in Kahane (1987), there are two extreme cases: (i) $A_n(t) \to A(t)$ in L_1 for each given t, in which case $A(t)$ is not almost surely zero and is said to be fully active (non-degenerate) on $\mathbb{R}_+$; (ii) $A_n(1)$ converges to 0 almost surely, in which case $A(t)$ is said to be degenerate on $\mathbb{R}_+$. Sufficient conditions for non-degeneracy and degeneracy in a general situation and relevant examples are provided in Kahane (1987).

The Rényi function, also known as the deterministic partition function, is defined for $t \in [0,1]$ as

$$\begin{aligned} R(q) &= \liminf_{n\to\infty} \frac{\log \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right)}{\log \left|I_k^{(n)}\right|} \\ &= \liminf_{n\to\infty} \left(-\frac{1}{n}\right) \log_2 \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right), \end{aligned}$$

where $I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}]$, $\quad k = 0, 1, \ldots, 2^n - 1$, $\left|I_k^{(n)}\right|$ is its length, and $\log_b$ is log to the base b.

Mannersalo *et al.* (2002) presented the conditions for L_2-convergence and scaling of moments:

<u>Theorem 1</u>

Suppose that the conditions **C1-C3** hold.

If, for some positive numbers δ and γ,

$$\exp\{-\delta|\tau|\} \leqslant \rho(\tau) = \frac{M(1,1;\tau) - 1}{M(2) - 1} \leqslant |C\tau|^{-\gamma}, \tag{4.4}$$

then $A_n(t)$ converges in L_2 if and only if

$$b > 1 + \sigma_\Lambda^2 = M(2).$$

If $A_n(t)$ converges in L_2, then the limit process $A(t)$ satisfies the recursion

$$A(t) = \frac{1}{b}\int_0^t \Lambda(s) d\tilde{A}(bs), \tag{4.5}$$

where the processes $\Lambda(t)$ and $\tilde{A}(t)$ are independent, and the processes $A(t)$ and $\tilde{A}(t)$ have identical finite-dimensional distributions.

If $A(t)$ is non-degenerate, the recursion (4.5) holds, $A(1) \in L_q$ for some $q > 0$, and $\sum_{n=0}^{\infty} c(q, b^{-n}) < \infty$, where $c(q,t) = \mathrm{E}\sup_{s\in[0,t]} |\Lambda^q(0) - \Lambda^q(s)|$, then there exist constants $\overline{C}$ and $\underline{C}$ such that

$$\underline{C} t^{q-\log_b \mathrm{E}\Lambda^q(t)} \leqslant \mathrm{E}A^q(t) \leqslant \overline{C} t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \tag{4.6}$$

which will be written as

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \ t \in [0,1].$$

If, on the other hand, $A(1) \in L_q$, $q > 1$, then the Rényi function is given by

$$R(q) = q - 1 - \log_b \mathrm{E}\Lambda^q(t) = q - 1 - \log_b M(q).$$

If $A(t)$ is non-degenerate, $A(1) \in L_2$, and $\Lambda(t)$ is positively correlated, then

$$\mathrm{Var}A(t) \geqslant \mathrm{Var}\int_0^t \Lambda(s) ds.$$

Hence, if $\int_0^t \Lambda(s)ds$ is strongly dependent, then $A(t)$ is also strongly dependent.

4.2 Ornstein-Uhlenbeck type processes

We recall some definitions and known results on Lévy processes (Skorokhod 1991, Bertoin 1996, Sato 1999, Kyprianou 2006) and Ornstein-Uhlenbeck type processes (Barndorff-Nielsen 2001, Barndorff-Nielsen and Shephard 2001) which are needed to construct a class of multifractal processes.

A random variable X is said to be infinitely divisible if its cumulant function has the Lévy-Khintchine form

$$C\{z;X\} = iaz - \frac{d}{2}z^2 + \int_{\mathbb{R}} \left(e^{izu} - 1 - izu\mathbf{1}_{[-1,1]}(u)\right)\nu(du), \tag{4.7}$$

where $a \in \mathbb{R}$, $d \geq 0$ and ν is the Lévy measure, that is, a non-negative measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}} \min\left(1, u^2\right)\nu(du) < \infty.$$

The triplet (a, d, ν) uniquely determines the random variable X. For a Gaussian random variable $X \sim \mathcal{N}(a, d)$, the Lévy triplet takes the form $(a, d, 0)$.

If X is self-decomposable, then there exists a stationary stochastic process $\{X(t), t \geq 0\}$, such that $X(t) \stackrel{d}{=} X$ and

$$X(t) = e^{-\lambda t}X(0) + \int_{(0,t]} e^{-\lambda(t-s)} d\grave{Z}(\lambda s) \tag{4.8}$$

for all $\lambda > 0$ (see Barndorff-Nielsen 1998). Conversely, if $\{X(t), t \geq 0\}$ is a stationary process and $\left\{\grave{Z}(t), t \geq 0\right\}$ is a Lévy process, independent of $X(0)$, such that $X(t)$ and $\grave{Z}(t)$ satisfy the Itô stochastic differential equation

$$dX(t) = -\lambda X(t)\,dt + d\grave{Z}(\lambda t) \tag{4.9}$$

for all $\lambda > 0$, then $X(t)$ is self-decomposable. A stationary process $X(t)$ of this kind is said to be an Ornstein-Uhlenbeck type process or an OU-type process, for short. The process $\grave{Z}(t)$ is termed the background driving Lévy process corresponding to the process $X(t)$. In fact (4.8) is the unique (up to indistinguishability) strong solution to Eq. (4.9) (Sato 1999).

If $X(t)$ is a square integrable OU process, it has the correlation function

$$r_X(t) = \exp\{-\lambda|t|\}, \quad t \in \mathbb{R}.$$

The following result is needed in the construction of mutifractal processes from OU-type processes:

Theorem 2

Let $X(t), t \in [0,1]$ be an OU type stationary process (4.8) such that the Lévy measure ν in (4.7) of the random variable $X(t)$ satisfies the condition that for some range of $q \in \mathbb{R}$,

$$\int_{|x|\geq 1} g_q(x)\nu(dx) < \infty,$$

where $g_q(x)$ denotes any of the functions $e^{2qx}, e^{qx}, e^{qx}|x|$. Then, for the geometric OU type process $\Lambda_q(t) := e^{qX(t)}$,

$$\sum_{n=0}^{\infty} c(q, b^{-n}) < \infty,$$

where $c(q,t) = \mathrm{E}\sup_{s\in[0,t]} |\Lambda_q(0)^q - \Lambda_q(s)^q|$.

The proof of Theorem 2 is given in Anh, Leonenko and Shieh (2008). To prove that a geometric OU-type process satisfies the covariance decay condition (4.4) in Theorem 1, the following proposition gives a general decay estimate which the driving Lévy processes Z in the next subsection indeed satisfy:

Consider the stationary OU-type process X defined by

$$dX(t) = -\lambda X(t)dt + dZ(\lambda t),$$

which has a stationary distribution $\pi(x)$ such that, for some $a > 0$,

$$\int |x|^a \, \pi(dx) < \infty. \tag{4.10}$$

Then there exist positive constants c and C such that

$$\mathrm{Cov}\left(e^{X(t)}, e^{X(0)}\right) \leq Ce^{-ct} \tag{4.11}$$

for all $t > 0$.

Masuda (2004) showed that, under the assumption (4.10), the stationary process $X(t)$ satisfies the $\beta-$mixing condition with coefficient $\beta_X(t) = O(e^{-ct}), t > 0$. Note that this is also true for the stationary process $e^{X(t)}$, since the $\sigma-$algebras generated by these two processes are equivalent. Hence,

$$\beta_{e^X}(t) = O(e^{-ct}), t > 0.$$

It then follows that

$$\text{Cov}\left(e^{X(t)}, e^{X(0)}\right) \leq const \times \beta_{e^X}(t) \leq Ce^{-ct}$$

(see Billingsley 1968).

4.3 Multifractal products of OU-type processes

In this section the results discussed in the previous sections are used to construct multifractal processes. The mother process of assumption **C1** will take the form

$$\Lambda(t) = \exp\left\{X\left(t\right) - c_X\right\}, \tag{4.12}$$

where $X\left(t\right)$ is a stationary OU type process (4.9) and c_X is a constant depending on the parameters of its marginal distribution such that $E\Lambda\left(t\right) = 1$.

All the definitions given in (4.1) - (4.3) and correspondingly all the statements of Theorem 1 are now understood to be in terms of the mother process (4.12). At this point however it is convenient to introduce separate notations for the moment generating function of Λ, which we denote by $M_\Lambda(\cdot)$, and the moment generating function of X, which we denote by $M(\cdot)$. Thus

$$M_\Lambda\left(z\right) = \mathrm{E}\exp\left(z\left(X\left(t\right) - c_X\right)\right) = \exp\{-zc_X\}M(z)$$

and

$$M_\Lambda\left(z_1, z_2; \left(t_1 - t_2\right)\right) = \mathrm{E}\exp\{z_1\left(X(t_1) - c_X\right) + z_2\left(X(t_2) - c_X\right)\}$$

$$= \exp\{-c_X (z_1 + z_2)\} M (z_1, z_2; (t_1 - t_2)) .$$

The correlation function of the mother process Λ then takes the form

$$\rho(\tau) = \frac{M_\Lambda(1,1;\tau) - 1}{M_\Lambda(2) - 1}.$$

The constant c_X (when it exists) can be obtained as

$$c_X = \log \mathrm{E} e^{X(t)} = \log M(1).$$

Accordingly, in view of (4.6), the Rényi function is obtained as

$$R(q) = q \left(1 + \frac{\log M(1)}{\log b} \right) - \frac{\log M(q)}{\log b} - 1.$$

Example (The log-gamma scenario)

We will use a stationary OU-type process with marginal gamma distribution $\Gamma(\beta, \alpha)$, which is self-decomposable, and, hence, infinitely divisible. The probability density function (pdf) of $X(t), t \in \mathbb{R}_+$, is given by

$$\pi(X) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} \mathbf{1}_{[0,\infty)}(x), \quad \alpha > 0, \beta > 0, \tag{4.13}$$

with the Lévy triplet of the form $(0, 0, \nu)$, where

$$\nu(du) = \frac{\beta e^{-\alpha u}}{u} \mathbf{1}_{[0,\infty)}(u) du,$$

while the Lévy process $\grave{Z}(t)$ in (4.9) is a compound Poisson subordinator

$$\grave{Z}(t) = \sum_{n=1}^{P(t)} Z_n,$$

with the $Z_n, n = 1, 2, \ldots,$ being independent copies of the random variable $\Gamma(1, \alpha)$ and $P(t), t \geq 0$, being a homogeneous Poisson process with intensity β. The logarithm of the characteristic function of $\grave{Z}(1)$ is

$$\kappa(z) = \log E e^{iz\grave{Z}(1)} = \frac{i\beta z}{\alpha - iz}, \quad z \in \mathbb{R},$$

and the (finite) Lévy measure $\tilde{\nu}$ of $\grave{Z}(1)$ is

$$\tilde{\nu}(du) = \alpha\beta e^{-\alpha u}\mathbf{1}_{[0,\infty)}(u)du.$$

C4 Consider a mother process of the form

$$\Lambda(t) = e^{X(t)-c_X} \text{ with } c_X = \log\frac{1}{(1-\frac{1}{\alpha})^{\beta}} \text{ and } \alpha > 1,$$

where $X(t), t \in \mathbb{R}_+$, is a stationary gamma OU-type stochastic process with marginal density (4.13) and covariance function

$$r_X(t) = \frac{\beta}{\alpha^2}e^{-\lambda|t|},\ t \in \mathbb{R}.$$

From the discussion above it follows that Theorem 1 and Theorem 2 and (4.11) can be applied to this setting to yield the following

Theorem 3

Suppose that condition **C4** holds, and let $Q = \{q : 0 < q < \alpha, \alpha > 2\}$. Then, for any $b > e^{-2c_X}(1-2\alpha)^{-\beta}$, $\beta > 0$, the stochastic processes $A_n(t)$ defined by (4.2) converge in L_2 to the stochastic process A(t) as $n \to \infty$ such that, if $A(1) \in L_q$ for $q \in Q$,

$$EA(t)^q \sim t^{R(q)+1},$$

where the Rényi function is given by

$$R(q) = q(1 + \frac{1}{\log b}\log\frac{1}{(1-\frac{1}{\alpha})^{\beta}}) + \frac{\beta}{\log b}\log(1-\frac{q}{\alpha}) - 1,\ q \in Q.$$

Example (The log-inverse Gaussian scenario)

We will use a stationary OU-type process with marginal inverse Gaussian distribution $IG(\delta, \gamma)$, which is self-decomposable and, hence, infinitely divisible. The pdf of $X(t), t \in \mathbb{R}_+$, is given by

$$\pi(x) = \frac{1}{\sqrt{2\pi}}\frac{\delta e^{\delta\gamma}}{x^{\frac{3}{2}}}e^{-(\frac{\delta^2}{x}+\gamma^2 x)\frac{1}{2}}\mathbf{1}_{[0,\infty)}(x),\ \delta > 0, \gamma \geq 0 \qquad (4.14)$$

with the Lévy triplet of the form $(0, 0, \nu)$, where

$$\nu(du) = \frac{1}{\sqrt{2\pi}} \frac{\delta}{u^{\frac{3}{2}}} e^{-\frac{\gamma^2 u}{2}} \mathbf{1}_{[0,\infty)}(u) du,$$

while the Lévy process $\grave{Z}(t)$ in (4.9) has the cumulant function

$$\kappa(z) = \log E e^{iz\grave{Z}(1)} = \frac{iz\delta}{\gamma\sqrt{1 - \frac{2iz}{\gamma^2}}}, \; z \in \mathbb{R}$$

that is, the Lévy triplet of $\grave{Z}(1)$ is of the form $(0, 0, \tilde{\nu})$, and $\grave{Z}(t)$ is the sum of two independent Lévy processes: $\grave{Z}(t) = \grave{Z}_1(t) + \grave{Z}_2(t)$. Here $\grave{Z}_1(t), t \in \mathbb{R}_+$, is an $IG(\frac{\delta}{2}, \gamma)$ subodinator with Lévy density

$$\tilde{\nu}_1(du) = \frac{1}{2\sqrt{2\pi}} \frac{\delta}{u\sqrt{u}} e^{-\frac{\gamma^2 u}{2}} \mathbf{1}_{[0,\infty)}(u) du,$$

which has infinitely many jumps in bounded time intervals, and $\grave{Z}_2(t), t \in \mathbb{R}_+$, is a compound Poisson subordinator:

$$\grave{Z}_2(t) = \frac{1}{\gamma^2} \sum_{n=1}^{P(t)} Z_n^2,$$

where the $Z_n, n = 1, 2, \ldots,$ are independent copies of the standard normal variable and $P(t), t \in \mathbb{R}_+$, is a homogeneous Poisson process with intensity $\frac{\delta\gamma}{2}$. The (finite) Lévy measure $\tilde{\nu}$ of $\grave{Z}_2(1)$ can be computed as

$$\tilde{\nu}_2(du) = \frac{1}{2\sqrt{2\pi}} \frac{\delta\gamma^2}{\sqrt{u}} e^{-\frac{\gamma^2 u}{2}} \mathbf{1}_{[0,\infty)}(u) du.$$

C5 Consider a mother process of the form

$$\Lambda(t) = e^{X(t) - c_X} \text{ with } c_X = \delta(\gamma - \sqrt{\gamma^2 - 2}) \text{ and } \gamma \geq \sqrt{2},$$

where $X(t), t \in \mathbb{R}_+$, is a stationary inverse Gaussian OU-type with marginal density (4.14) and covariance function

$$r_X(t) = \frac{\delta}{\gamma^3} e^{-\lambda|t|}, \; t \in \mathbb{R}.$$

From the discussion above it follows that Theorem 1 and Theorem 2 and (4.11) can be applied to this setting to yield the following

Theorem 4

Suppose that condition **C5** holds, and let $Q = \{q : 0 < q < \frac{\gamma^2}{2}, \alpha > 2\}$. Then, for any $b > e^{-2c_X + \delta(\gamma - \sqrt{\gamma^2 - 4})}$, the stochastic processes $A_n(t)$ defined by (4.2) converge in L_2 to the stochastic process A(t) as $n \to \infty$ such that, if $A(1) \in L_q$ for $q \in Q$,

$$EA(t)^q \sim t^{R(q)+1},$$

where the Rényi function is given by

$$R(q) = q(1 + \frac{\delta(\gamma - \sqrt{\gamma^2 - 2})}{\log b}) + \frac{\delta}{\log b}\sqrt{\gamma^2 - 2q} - \frac{\gamma\delta}{\log b} - 1,\ q \in Q.$$

Example (The log-spectrally negative α-stable scenario)

We propose a stationary OU-type process satisfying the Itô stochastic differential equation

$$dX\,(t) = -\lambda X\,(t)\,dt + d\grave{Z}\,(\lambda t)\,,$$

for all $\lambda > 0$, where $\{Z_t, t \geq 0\}$ is a càdlàg spectrally negative α-stable process with $1 < \alpha < 2$ and stationary and independent increments.

Due to the absence of positive jumps, Patie (2007) states that it is possible to extend the characteristic exponent of $\{Z_t\}$ on the negative imaginary line to derive its Laplace exponent, $\psi(z) = Ee^{-zZ(t)} = u^\alpha,\ u \geq 0$. However, as we are interested in the case where there is an absence of negative jumps, the logarithm of the characteristic function of $Z(1)$ is

$$\kappa_{Z(1)}(z) = \log Ee^{izZ(1)} = (iz)^\alpha,$$

and the (finite) Lévy measure $\tilde{\nu}$ of $Z(1)$

$$\tilde{\nu}(du) = cu^{-\alpha-1}\mathbf{1}_{(0,\infty)}(u)du,\ c > 0.$$

The related logarithm of the characteristic function of X is

$$\kappa_X(z) = \frac{1}{\lambda}\int_0^z \frac{\kappa_{Z(1)}(\xi)}{\xi}\, d\xi = \frac{(iz)^\alpha}{\alpha\lambda}.$$

C1 Consider a mother process of the form

$$\Lambda(t) = e^{X(t)-c_X} \text{ with } c_X = \frac{1}{\alpha\lambda},$$

where $X(t), t \in \mathbb{R}$, is a stationary spectrally negative α-stable OU-type stochastic process.

All conditions hold for Theorems 1 and 2, so we can now formulate the following:

<u>Theorem 5</u>

Suppose that condition **C1** holds, and let $Q = \{q : q > 0\}$. Then, for any $b > e^{-\frac{2^\alpha}{\alpha\lambda}-\frac{2}{\alpha\lambda}}$, $\lambda > 0$, the stochastic processes $A_n(t)$ converge in L_2 to the stochastic process A(t) as $n \to \infty$ such that, if $A(1) \in L_q$ for $q \in Q$,

$$EA(t)^q \sim t^{R(q)+1},$$

where the Rényi function is given by

$$R(q) = q(1 + \frac{1}{\log b}\frac{1}{\alpha\lambda}) - \frac{1}{\log b}\frac{q^\alpha}{\alpha\lambda} - 1,\ q \in Q.$$

In *Table 1*, all corresponding Rènyi functions, and ranges of q for the L_2-convergence of A_n to A for the models discussed in this paper, are provided. For further scenarios and tables for ready reference see Anh et al (2008, 2010).

For $q \in Q \cap [1, 2]$, the condition $A(1) \in L_q, q > 1$ follows from the L_2 convergence; thus the above results hold at least for this range. For q outside this range, the condition is still to be verified for the validity of multifractal moment scaling. However, Anh et al (2010) illustrates that through simulation experiments, convergence to multifractality should hold for values of q larger than 2. Hence there is scope for relaxing the condition $A(1) \in L_q$ for $q = 1, 2$.

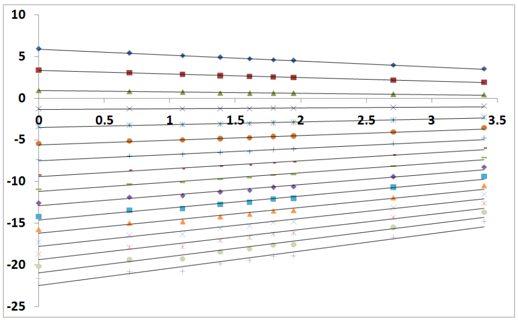
Scenario	Rényi function
Log-Γ	$\tau(q) = q(1 + \frac{1}{\log b}\log\frac{1}{(1-\frac{1}{\alpha})^{\beta}}) + \frac{\beta}{\log b}\log(1-\frac{q}{\alpha}) - 1,$ $\alpha > 0,\ \beta > 0,\ b > (1-\frac{1}{\alpha})^{2\beta}(1-2\alpha)^{-\beta},$ $q \in Q = \{q : 0 < q < \alpha, \alpha > 2\}$
Log-IG	$\tau(q) = q(1 + \frac{\delta(\gamma-\sqrt{\gamma^2-2})}{\log b}) + \frac{\delta}{\log b}\sqrt{\gamma^2-2q} - \frac{\gamma\delta}{\log b} - 1,$ $\delta > 0,\ \gamma > 0,\ b > e^{-\delta\gamma+2\delta\sqrt{\gamma^2-2}-\delta\sqrt{\gamma^2-4}},$ $q \in Q = \{q : 0 < q < \frac{\gamma^2}{2}, \gamma \geq 2\}$
Log-NIG	$\tau(q) = q(1 + \frac{\delta\sqrt{\alpha^2-\beta^2}}{\log b} - \frac{\delta\sqrt{\alpha^2-(1+\beta)^2}}{\log b}) + \frac{\delta\sqrt{\alpha^2-(q-\beta)^2}}{\log b} - \frac{\delta\sqrt{\alpha^2-\beta^2}}{\log b} - 1,$ $\mu \in \mathbb{R},\ 0 \leq \lvert\beta\rvert \leq \alpha,\ \delta > 0,\ b > e^{-\delta\sqrt{\alpha^2-\beta^2}+2\delta\sqrt{\alpha^2-(\beta+1)^2}-\delta\sqrt{\alpha^2-(\beta+2)^2}},$ $q \in Q = \{q : 0 < q < \alpha - \lvert\beta\rvert, \lvert\beta+2\rvert < \alpha\}$
Log-TS	$\tau(q) = q(1 + \frac{\delta(\gamma-(\gamma^{\frac{1}{\kappa}}-2)^{\kappa})}{\log b}) + \frac{\delta}{\log b}(\gamma^{\frac{1}{\kappa}} - 2q)^{\kappa} - \frac{\gamma\delta}{\log b} - 1,$ $0 < \kappa < 1,\ \delta > 0,\ \gamma \geq 0,\ b > e^{2\delta(\gamma^{\frac{1}{\kappa}}-2)^{\kappa}-\delta(\gamma^{\frac{1}{\kappa}}-4)^{\kappa}-\delta\gamma},$ $q \in Q = \{q : 0 < q < \frac{\gamma^{\frac{1}{\kappa}}}{2}, \gamma \geq 4^{\kappa}\}$
Log-SNS	$\tau(q) = q(1 + \frac{1}{\log b}\frac{1}{\alpha\lambda}) - \frac{1}{\log b}\frac{q^{\alpha}}{\alpha\lambda} - 1,$ $\lambda > 0,\ b > e^{-\frac{2^{\alpha}}{\alpha\lambda}-\frac{2}{\alpha\lambda}},$ $q \in Q = \{q : q > 0\}$

Table 1: Log-distribution scenarios for multifractal products of stationary OU-type processes

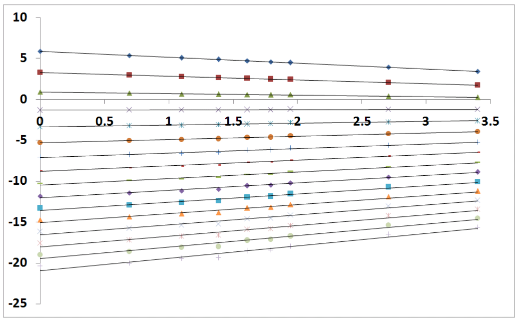
5 Data fitting

Next we want to show some evidence of multifractality in real financial data through the simple empirical test outlined in the previous section. This will become the motivation of the paper enabling us to look at various constructions of multifractal processes which can be implemented into the risky asset model (3.1).

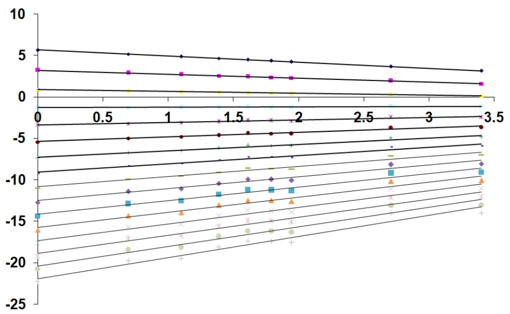
To estimate the scaling function, Calvet and Fisher (2002) proposed a method based on a partition function. It allowed them to successfully detect the multifractal properties of real financial data (in their case, the French stock market). In this paper we will use data sets of exchange rates between USD and DM(T=6333), FF(T=6428), GBP(T=4510) and JY(T=4510) where an observation has been taken at the close of every working day over various periods of time between 1971 and 2001. Although the FF and the DM are no longer in trade they allow us to test our models on quite different behaviours as these series do present a high degree of multifractality.



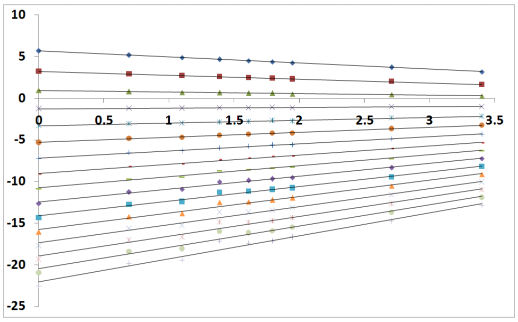
(a) DM



(b) FF



(c) GBP



(d) JY

Figure 1: The partition function

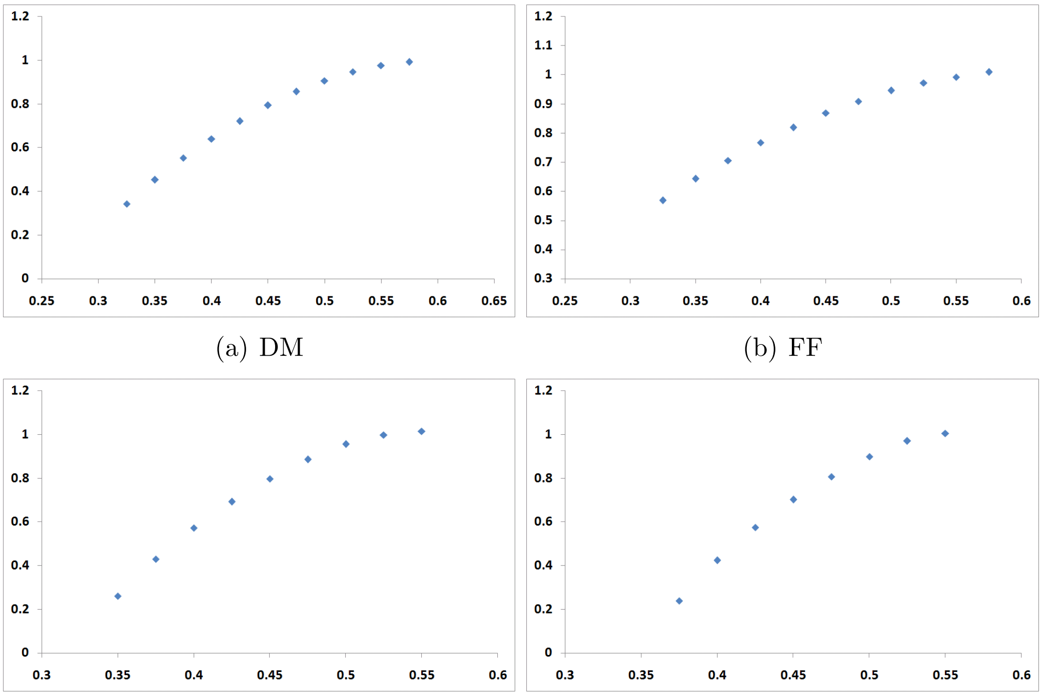
The partition function is denoted $\pi_\delta(Y, q)$, and defined by partitioning the series {Y(t)} into n subintervals of length δ for each moment q

$$\pi_\delta(Y, q) = \sum_{i=1}^{n} |Y_{\lceil i\delta \rceil} - Y_{\lceil (i-1)\delta \rceil}|^q, \tag{5.1}$$

where $\lceil \cdot \rceil$ is the integer part (ceiling) operator. By allowing this partition function to be the empirical counterpart of $E(|Y(t)|^q)$ in (3.2), we have

$$\log \pi_\delta(Y, q) = \tau_A(\frac{q}{2}) \log \delta + \log T + const,$$

where $T = n\delta$ and $const = \log c_A(\frac{q}{2})(\sqrt{2\sigma^2})^q \Gamma(\frac{1+q}{2})/\sqrt{\pi}$. Thus by plotting $\log \pi_\delta(Y, q)$ against $\log \delta$ for various moments q (*Figure 1*), we can obtain $\hat{\tau}_A(\frac{q}{2})$.



(a) DM

(b) FF

(c) GBP

(d) JY

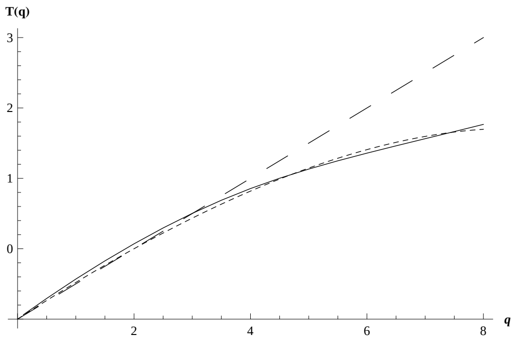
Figure 2: Estimation of the multifractal spectrum

In addition, the multifractal spectrum (see *Figure 2*) is estimated by

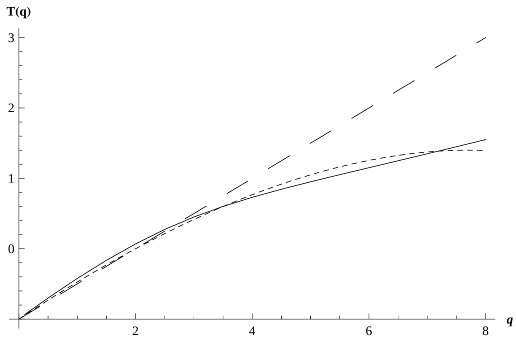
$$\hat{f}(\alpha) = \inf_q [q\alpha - \hat{\tau}(q)].$$

Overall, *Figure 1* and *Figure 2*, based on non-parametric estimates, show clear evidence of multifractality in the data.

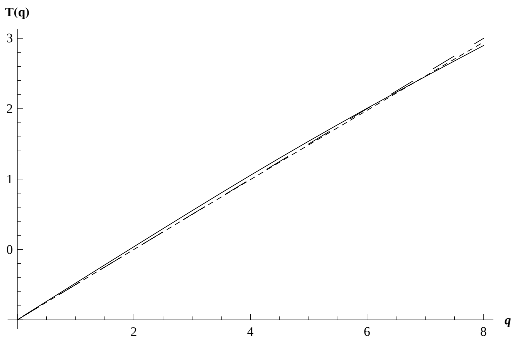
To judge about the applicability of the models discussed, we will now compare the non-parametric estimate of the scaling function $\hat{\tau}(q)$ with the Rényi function obtained for the scenarios corresponding to the gamma, inverse Gaussian, normal inverse Gaussian, tempered stable, and spectrally negative stable distributions (see *Table 1*).



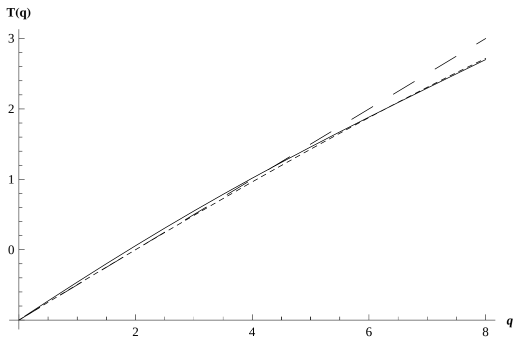
(a) DM ($\hat{\alpha} = 10.02, \hat{\beta} = 1.62, \hat{b} = 1.12$)



(b) FF ($\hat{\alpha} = 12.11, \hat{\beta} = 2.64, \hat{b} = 1.10$)



(c) GBP ($\hat{\alpha} = 8.01, \hat{\beta} = 0.40, \hat{b} = 3.30$)

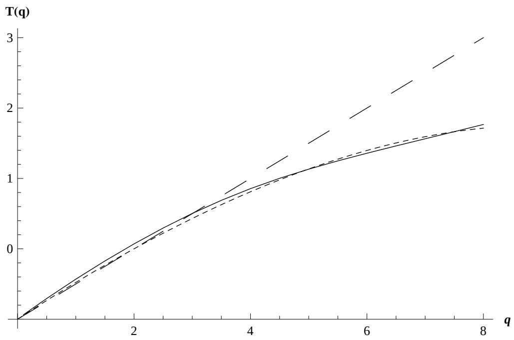


(d) JY ($\hat{\alpha} = 8.01, \hat{\beta} = 1.00, \hat{b} = 2.64$)

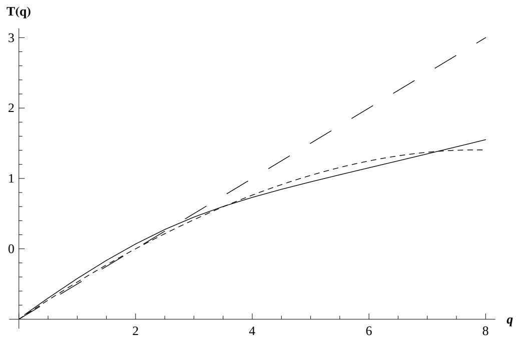
Figure 3: Log-gamma scenario of multifractal products of geometric OU-type processes: Solid (line)- non-parametric estimate of $\tau(q)$, Thick-dashed- fitted parametric estimate of $\tau(q)$, Sparse-dashed- Brownian motion case.

For all remaining empirical work, the values of the increment δ in the calculation of the partition function (5.1) have been set at 1,2,3,4,5,6,7,15 and 30 (i.e. short to medium term approximately), together with the values of moment q ranging from 0 to 8 by 0.5 increments. This enables us to compute $\hat{\tau}(q)$ for every q and for each data set.

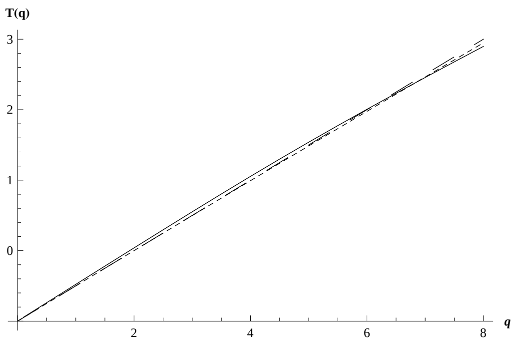
All parameters of the parametric Rényi functions including the scaling parameter b and parameters of the marginal distribution of $X(t)$ were estimated



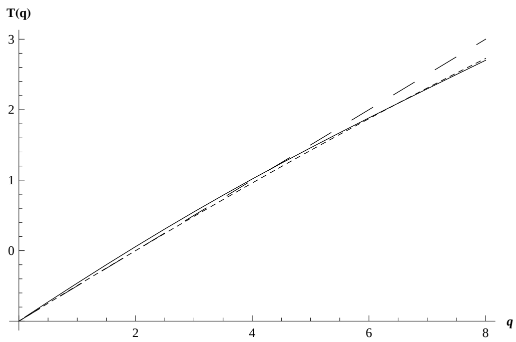
(a) DM ($\hat{\delta} = 0.68, \hat{\gamma} = 4.61, \hat{b} = 1.04$)



(b) FF ($\hat{\delta} = 0.93, \hat{\gamma} = 4.72, \hat{b} = 1.04$)



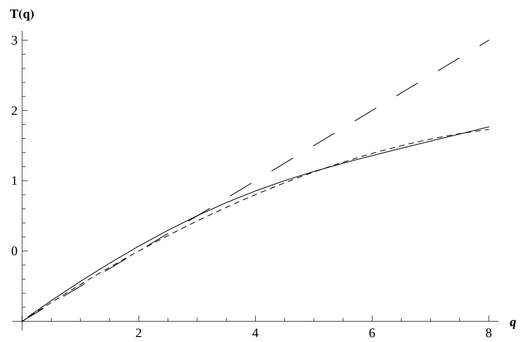
(c) GBP ($\hat{\delta} = 0.32, \hat{\gamma} = 4.01, \hat{b} = 2.31$)



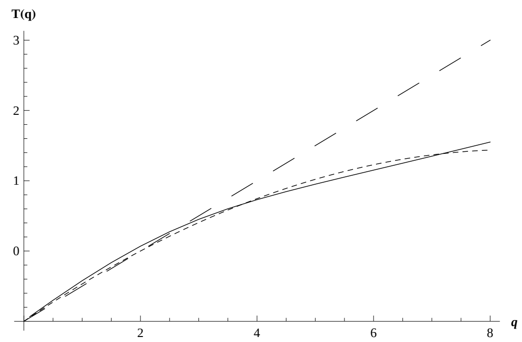
(d) JY ($\hat{\delta} = 1.38, \hat{\gamma} = 4.52, \hat{b} = 1.54$)

Figure 4: Log-inverse Gaussian scenario of multifractal products of geometric OU-type processes: Solid (line)- non-parametric estimate of $\tau(q)$, Thick-dashed- fitted parametric estimate of $\tau(q)$, Sparse-dashed- Brownian motion case.

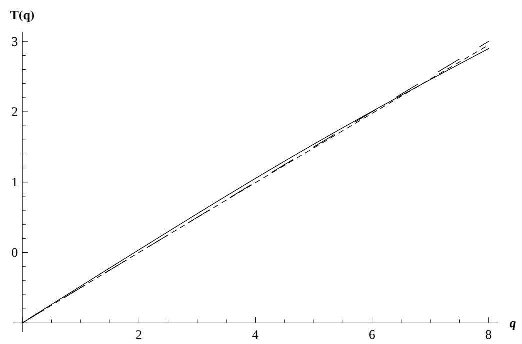
using non-linear least squares. Our aim is to minimise the mean square error between the scaling function estimated from the data and the corresponding analytical forms; the data-fitted Renyi function will be denoted by $\tau_{\hat{\theta}}(q)$. For other techniques of estimation which could be adapted here see Taufer and Leonenko (2009) and Taufer *et al.* (2011) which apply characteristic function estimation techniques to OU processes and OU-based stochastic volatility models.



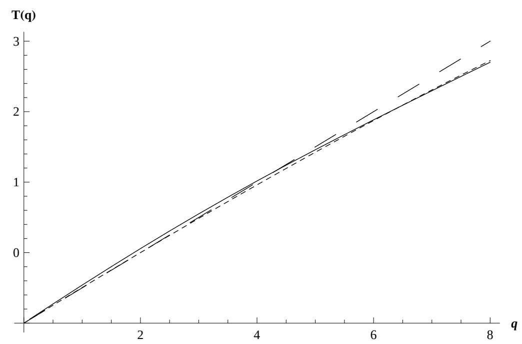
(a) DM ($\hat{\alpha} = 10.98, \hat{\delta} = 1.31, \hat{b} = 1.94, \hat{\beta} = 1.74$)



(b) FF ($\hat{\alpha} = 14.20, \hat{\delta} = 2.76, \hat{b} = 2.17, \hat{\beta} = 0.46$)



(c) GBP ($\hat{\alpha} = 44.87, \hat{\delta} = 0.05, \hat{b} = 2.61, \hat{\beta} = 36.87$)

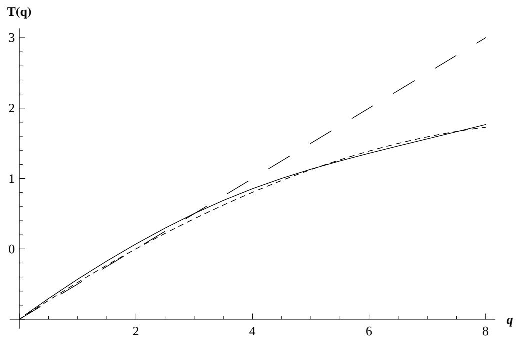


(d) JY ($\hat{\alpha} = 36.01, \hat{\delta} = 0.38, \hat{b} = 3.72, \hat{\beta} = 28.01$)

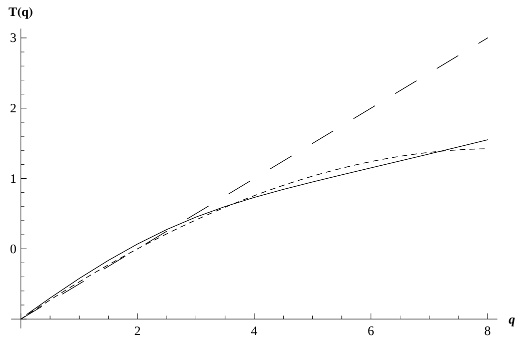
Figure 5: Log-normal inverse Gaussian scenario of multifractal products of geometric OU-type processes: Solid (line)- non-parametric estimate of $\tau(q)$, Thick-dashed- fitted parametric estimate of $\tau(q)$, Sparse-dashed- Brownian motion case.

Figures 3-7 report, for each scenario of *Table 1*, the non-parametric estimate

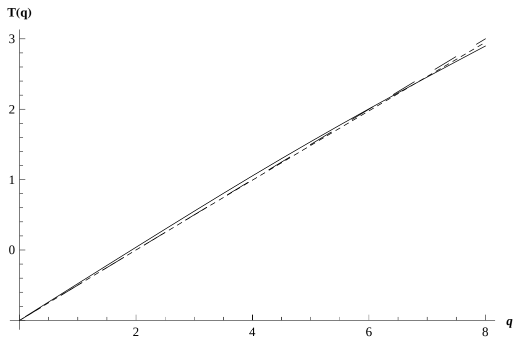
$\hat{\tau}(q)$ and the model-based one fitted from the data $\tau_{\hat{\theta}}(q)$. As we see the DM and the FF do show quite a different behaviour from the other currencies (which tend to be less liquid). All fitted scenarios seem to be able to capture quite well the behaviour of the non-parametic estimate $\hat{\tau}(q)$, with some *distinguo*. It appears that the most difficult series to fit is the FF whose curvature of $\hat{\tau}(q)$, especially for $q > 4$, is quite difficult to be captured by any of the scenarios except the Log-SNS one which looks quite apt for the exchange-rate problem as it obtains good results in all cases.



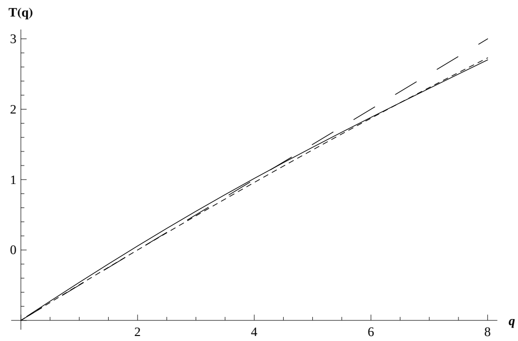
(a) DM ($\hat{\delta} = 1.56, \hat{\gamma} = 16.97, \hat{\kappa} = 0.89, \hat{b} = 1.10$)



(b) FF ($\hat{\delta} = 11.85, \hat{\gamma} = 4.01, \hat{\kappa} = 0.22, \hat{b} = 1.01$)

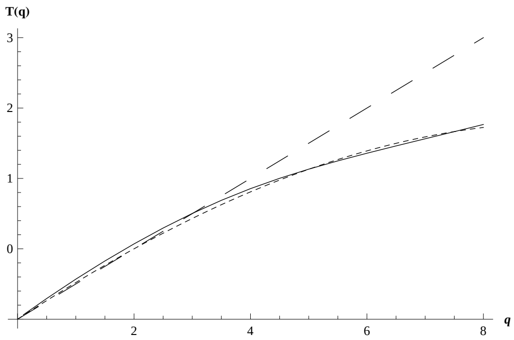


(c) GBP ($\hat{\delta} = 0.56, \hat{\gamma} = 4.07, \hat{\kappa} = 0.63, \hat{b} = 1.98$)

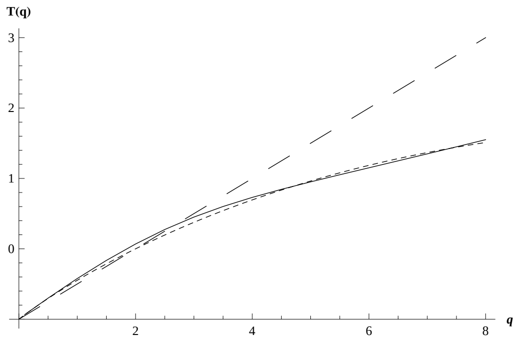


(d) JY ($\hat{\delta} = 0.26, \hat{\gamma} = 4.01, \hat{\kappa} = 0.67, \hat{b} = 3.90$)

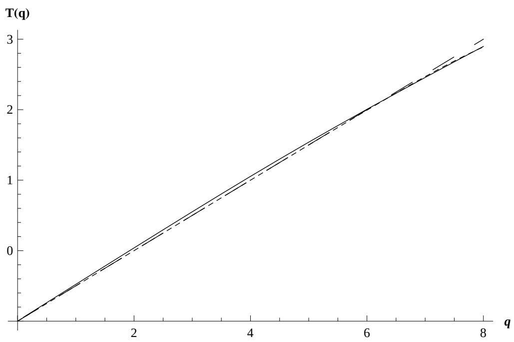
Figure 6: Log-tempered stable scenario of multifractal products of geometric OU-type processes: Solid (line)- non-parametric estimate of $\tau(q)$, Thick-dashed- fitted parametric estimate of $\tau(q)$, Sparse-dashed- Brownian motion case.



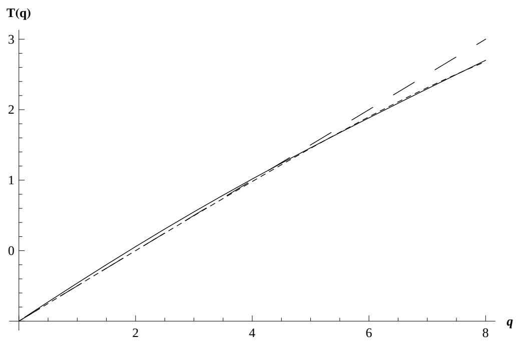
(a) DM ($\hat{\alpha} = 2.82, \hat{\lambda} = 7.08, \hat{b} = 10.63$)



(b) FF ($\hat{\alpha} = 1.94, \hat{\lambda} = 67.73, \hat{b} = 1.05$)



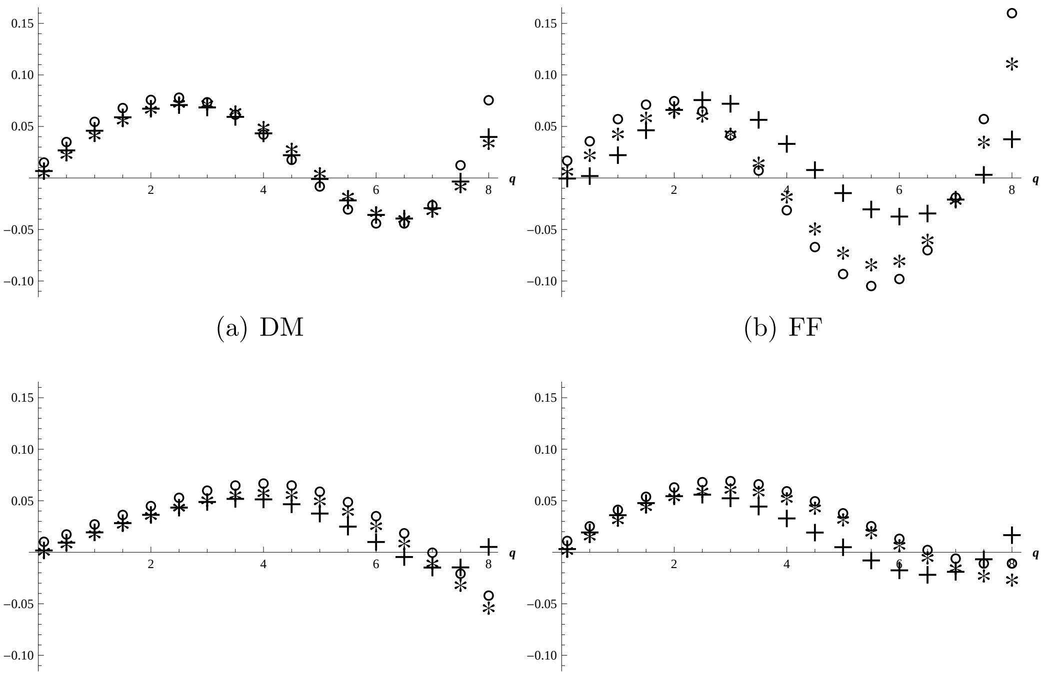
(c) GBP ($\hat{\alpha} = 3.64, \hat{\lambda} = 42.11, \hat{b} = 8.10$)



(d) JY ($\hat{\alpha} = 3.86, \hat{\lambda} = 68.35, \hat{b} = 30.86$)

Figure 7: Log-spectrally negative α-stable scenario of multifractal products of geometric OU-type processes: Solid (line)- non-parametric estimate of $\tau(q)$, Thick-dashed- fitted parametric estimate of $\tau(q)$, Sparse-dashed- Brownian motion case.

To better appreciate the details, the residuals $\hat{\tau}(q) - \tau_{\hat{\theta}}(q)$ for selected scenarios (other cases presented similar results) are depicted in *Figure 8*. The scale is the same in all graphs to allow visual comparison across all currencies. As we see the Log-SNS scenario is the closest to the curvature of $\hat{\tau}(q)$ notwithstanding some departures are still present. These impressions are confirmed by the results in *Table 2* where the residual sum of squares after regression (RSS) are reported for all cases. According to this criterion the Log-SNS scenario always outperform the others, especially in the case of the FF.



(a) DM (b) FF

(c) GBP (d) JY

Figure 8: The residual plots after regression for some selected models: Cross- Spectrally negative α stable; Star- Normal Inverse Gaussian; Circle- Gamma.

Recall that the inverse gaussian distribution is a special case of the tempered stable one when κ is set to 1/2. We note that in the case of the DM, GBP and JY the greater flexibility given by the additional parameter appears to be offset by the more complex estimation problem as the RSS's are all quite

Scenario	Residual Sum of Squares			
	DM	FF	GBP	JY
Log-Γ	0.0387	0.0896	0.0275	0.0275
Log-IG	0.0359	0.0810	0.0279	0.0279
Log-NIG	0.0357	0.0550	0.0280	0.0280
Log-TS	0.0353	0.0682	0.0283	0.0285
Log-SNS	0.0343	0.0294	0.0191	0.0191

Table 2: The residual sum of squares after regression

similar. However, for the FF, which appears as the most difficult case, the tempered stable scenario clearly outperform the inverse gaussian one.

6 Conclusion

We have reviewed a class of models based on multifractal activity time and have tested their flexibility in applications through the use of exchange rate data. Multifratal processes based on products of geometric OU processes appear well apt for applications in different fields as several different mother processes for their construction are available; properties of different scenarios are easily derived by the characteristic function of the underlying mother process.

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