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The Generalized Gini Welfare Function in the Framework of Symmetric Choquet Integration

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Abstract

In the context of Social Welfare and Choquet integration, we briefly review the classical Gini inequality index for populations of $n \geq 2$ individuals, including the associated Lorenz area formula, plus the k -additivity framework for Choquet integration introduced by Grabisch, particularly in the additive and 2-additive symmetric cases. We then show that any 2-additive symmetric Choquet integral can be written as the difference between the arithmetic mean and a multiple of the classical Gini inequality index, with a given interval constraint on the multiplicative parameter. In the special case of positive parameter values, this result corresponds to the well-known Ben Porath and Gilboa's formula for Weymark's generalized Gini welfare functions, with linearly decreasing (inequality averse) weight distributions.

Keywords: Social Welfare, Gini Inequality Index, Symmetric Capacities and Choquet Integrals, OWA Functions, 2-Additivity and Equidistant Weights.

1 Introduction

The Gini inequality index [24, 25, 21, 15] plays a crucial role in Social Welfare Theory and the measurement of economic inequality [2, 45]. In the literature several extensions of the Gini index have been proposed [14, 47, 48, 49, 16, 9, 4], in particular the generalized Gini inequality index and the associated welfare function introduced by Weymark [47] on the basis of Blackorby and Donaldson's correspondence formula [5, 6],

$$A_G(\mathbf{x}) = \bar{x} - G_A(\mathbf{x})$$

where $G_A(\mathbf{x})$ denotes the (absolute) Gini inequality index, $A_G(\mathbf{x})$ is the associated generalized Gini welfare function, and $\mathbf{x} = (x_1, \dots, x_n)$ represents the

income distribution of a population of $n \geq 2$ individuals. Recently, the extended interpretation of this formula in terms of the dual decomposition [19] of aggregation functions has been discussed in [20, 1].

The generalized Gini welfare functions introduced by Weymark have the form

$$A(\mathbf{x}) = \sum_{i=1}^n w_i x_{(i)}$$

where $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. These welfare functions correspond to the ordered weighted averaging (OWA) functions introduced by Yager [50], which in turn correspond (see [17]) to the symmetric Choquet integrals. Moreover, the principle of inequality aversion for welfare functions requires non-increasing weights $1 \geq w_1 \geq w_2 \geq \dots \geq w_n \geq 0$, with $\sum_{i=1}^n w_i = 1$.

The use of non-additivity and Choquet integration [12] in Social Welfare and Decision Theory dates back to the seminal work of Schmeidler [43, 44], Ben Porath and Gilboa [3], and Gilboa and Schmeidler [22, 23]. In the discrete case, Choquet integration [41, 10, 13, 26, 27, 36] corresponds to a generalization of both weighted averaging and ordered weighted averaging, which remain as special cases. For recent reviews of Choquet integration see [32, 35, 33, 34].

The complex structure of Choquet capacities can be described in the k -additivity framework introduced by Grabisch [28, 30, 29, 7, 8, 40]. The 2-additive case, in particular, has been examined in [40, 37, 38]. Due to its low complexity and versatility it is relevant in a variety of modelling contexts.

The characterization of symmetric Choquet integrals (OWA functions) has been studied in [7, 8, 18, 40]. It is shown that in the k -additive case the generating function of the OWA weights is polynomial of degree $k - 1$. In the symmetric 2-additive case, in particular, the generating function is linear and thus the weights are equidistant, in analogy with the classical Gini welfare function.

In this paper we examine explicitly the family of symmetric Choquet integrals (OWA functions) of the 2-additive type and show that, in fact, any 2-additive OWA function can be written as the difference between the arithmetic mean and a multiple of the classical Gini inequality index, with a given interval constraint on the multiplicative parameter. In the special case of positive parameter values, this result corresponds to the well-known Ben Porath and Gilboa's formula [3] for Weymark's generalized Gini welfare functions with linearly decreasing (inequality averse) weight distributions.

The paper is organized as follows. In Section 2 we review the classical Gini index for populations of $n \geq 2$ individuals, including the Lorenz area formula in the discrete case. In Section 3 we present the basic definitions and results on capacities and Choquet integrals, particularly in the additive and 2-additive cases. In Sections 4 and 5 we consider symmetric Choquet integration and we present the main result of the paper, concerning the parametric expression of the 2-additive OWA functions in terms of the arithmetic mean and a multiple of the classical Gini inequality index.

2 Gini Inequality Index and Welfare Function

Consider a population of $n \geq 2$ individuals whose income distribution is represented by $\mathbf{x} = (x_1, \dots, x_n)$. Typically the range of the income values is taken

to be $[0, \infty)$ but in this paper, apart from the derivation of the Lorenz area formula below, it could be the whole real line.

We define the (*absolute*) *classical Gini inequality index* as

$$G_A^c(\mathbf{x}) = - \sum_{i=1}^n \frac{n-2i+1}{n^2} x_{(i)} \quad (1)$$

where $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. This expression shows explicitly the coefficients of the ordered income variables and is the most convenient in our presentation. In what follows we will omit "classical" and refer only to "Gini inequality index".

The traditional form of the Gini inequality index $G_A^c(\mathbf{x})$ is given by

$$G_A^c(\mathbf{x}) = \frac{1}{2n^2} \sum_{i,j=1}^n |x_i - x_j| \quad (2)$$

which can be easily shown to be equivalent to (1). In fact, the double summation expression for $n^2 G_A^c(\mathbf{x})$ as in (2) corresponds to

$$\begin{aligned} & (x_{(n)} - x_{(n-1)}) + (x_{(n)} - x_{(n-2)}) + \dots + (x_{(n)} - x_{(2)}) + (x_{(n)} - x_{(1)}) \\ & \quad + (x_{(n-1)} - x_{(n-2)}) + \dots + (x_{(n-1)} - x_{(2)}) + (x_{(n-1)} - x_{(1)}) \\ & \quad \quad \quad \vdots \\ & \quad \quad \quad + (x_{(3)} - x_{(2)}) + (x_{(3)} - x_{(1)}) \\ & \quad \quad \quad \quad + (x_{(2)} - x_{(1)}) \end{aligned}$$

which can be rewritten as

$$(n-1)x_{(n)} + ((n-2)-1)x_{(n-1)} + \dots + (1-(n-2))x_{(2)} + (-(n-1))x_{(1)}. \quad (3)$$

It follows that

$$n^2 G_A^c(\mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^n |x_i - x_j| = - \sum_{i=1}^n (n-2i+1)x_{(i)}. \quad (4)$$

In the discrete case, the Lorenz area formula can be derived as follows. Consider

$$V(\mathbf{x}) = \sum_{i=1}^n (x_{(1)} + \dots + x_{(i)}) = nx_{(1)} + (n-1)x_{(2)} + \dots + x_{(n)} \quad (5)$$

$$U(\mathbf{x}) = \sum_{i=1}^n (x_{(i)} + \dots + x_{(n)}) = x_{(1)} + 2x_{(2)} + \dots + nx_{(n)}. \quad (6)$$

We can easily express $U(\mathbf{x})$ in terms of $V(\mathbf{x})$,

$$\begin{aligned} U(\mathbf{x}) &= \sum_{i=1}^n (x_{(i)} + \dots + x_{(n)}) \\ &= \sum_{i=1}^n [(x_{(1)} + \dots + x_{(n)}) - (x_{(1)} + \dots + x_{(i)}) + x_{(i)}] \\ &= n^2 \bar{x} - V(\mathbf{x}) + n\bar{x} = n(n+1)\bar{x} - V(\mathbf{x}) \end{aligned} \quad (7)$$

where $\bar{x} = (x_{(1)} + \dots + x_{(n)})/n$. Since

$$\begin{aligned} n^2 G_A^c(\mathbf{x}) &= -\sum_{i=1}^n (n-2i+1)x_{(i)} \\ &= -((n-1)x_{(1)} + (n-3)x_{(2)} + \dots + (-n+1)x_{(n)}) \end{aligned} \quad (8)$$

we can write $G_A^c(\mathbf{x})$ in terms of \bar{x} and $V(\mathbf{x})$,

$$n^2 G_A^c(\mathbf{x}) = -(V(\mathbf{x}) - U(\mathbf{x})) = n(n+1)\bar{x} - 2V(\mathbf{x}). \quad (9)$$

Consider now the area illustrated in Fig. 1. The diagonal line and the Lorenz “curve” are hypothetical and are indicated only to suggest the analogy with the continuous case. In the discrete case we have just the vertical differences

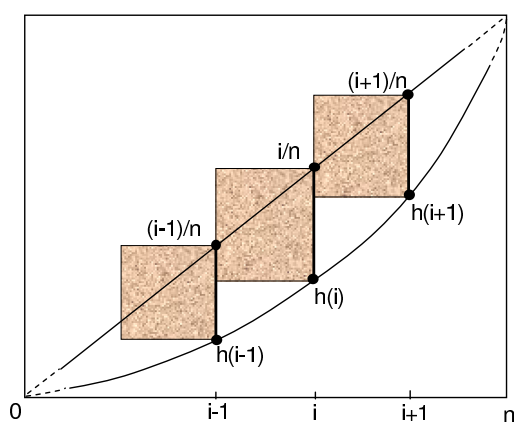


Figure 1: Lorenz area in the discrete case.

between the diagonal i/n values, associated with uniform cumulative income distribution, and the actual cumulative income distribution expressed by the $h(i)$ values,

$$h(i) = \frac{x_{(1)} + \dots + x_{(i)}}{x_{(1)} + \dots + x_{(n)}} \quad (10)$$

where we assume $x_{(i)} \geq 0$ for $i = 1, \dots, n$ and $x_{(n)} > 0$, so that $\bar{x} > 0$.

The total area H in Fig. 1 is therefore given by

$$\begin{aligned} H &= \sum_{i=1}^n \left(\frac{i}{n} - h(i) \right) = \sum_{i=1}^n \left(\frac{i}{n} - \frac{x_{(1)} + \dots + x_{(i)}}{x_{(1)} + \dots + x_{(n)}} \right) \\ &= \frac{1}{n\bar{x}} \left[\sum_{i=1}^n (i\bar{x} - (x_{(1)} + \dots + x_{(i)})) \right] \\ &= \frac{1}{n\bar{x}} \left[\frac{n(n+1)}{2} \bar{x} - V(\mathbf{x}) \right] \\ &= \frac{1}{n\bar{x}} \left[\frac{n^2}{2} G_A^c(\mathbf{x}) \right] = \frac{n}{2\bar{x}} G_A^c(\mathbf{x}). \end{aligned} \quad (11)$$

Finally, we obtain

$$G_A^c(\mathbf{x}) = \frac{H}{n/2} \bar{x} \quad (12)$$

where the Lorenz area $H/(n/2)$ corresponds to the *relative Gini inequality index*.

The *welfare function* associated with the classical Gini inequality index is

$$A_G^c(\mathbf{x}) = \bar{x} - G_A^c(\mathbf{x}) \quad (13)$$

and it can be written as

$$A_G^c(\mathbf{x}) = \sum_{i=1}^n \frac{2(n-i)+1}{n^2} x_{(i)} = \sum_{i=1}^n \frac{1}{n} x_{(i)} + \sum_{i=1}^n \frac{n-2i+1}{n^2} x_{(i)} \quad (14)$$

where the coefficients of the Gini index sum up to zero, $\sum_{i=1}^n (n-2i+1) = 0$.

3 Capacities and Choquet Integrals

In this section we present a brief review of the basic facts on Choquet integration, focusing on the additive and 2-additive cases as described by their Möbius representations. For recent reviews on Choquet integration see [32, 35, 33, 34] for the general case, and [37, 38] for the 2-additive case.

Consider a finite set of interacting criteria $N = \{1, 2, \dots, n\}$. The subsets $S, T \subseteq N$ are usually called coalitions.

Definition 1 A capacity [12] on the set N is a set function $\mu : 2^N \rightarrow [0, 1]$ satisfying

- (i) $\mu(\emptyset) = 0, \mu(N) = 1$ (boundary conditions)
- (ii) $S \subseteq T \subseteq N \Rightarrow \mu(S) \leq \mu(T)$ (monotonicity).

Capacities are also known as *fuzzy measures* [46] or *non-additive measures* [13]. Given two coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$, the capacity μ is said to be

- additive for S, T if $\mu(S \cup T) = \mu(S) + \mu(T)$,
- subadditive for S, T if $\mu(S \cup T) < \mu(S) + \mu(T)$,
- superadditive for S, T if $\mu(S \cup T) > \mu(S) + \mu(T)$.

If any of these properties holds for all coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$, the capacity μ is said to be additive, subadditive, or superadditive, respectively. In the additive case, in particular, we have $\mu(N) = \mu(\bigcup_{i=1}^n \{i\}) = \sum_{i=1}^n \mu(\{i\}) = 1$.

Definition 2 Let μ be a capacity on N . The Choquet integral [12, 26, 27] of a point $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ with respect to μ is defined as

$$C_\mu(\mathbf{x}) = \sum_{i=1}^n [\mu(A_{(i)}) - \mu(A_{(i+1)})] x_{(i)} \quad (15)$$

where (\cdot) indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \dots, (n)\}$ and $A_{(n+1)} = \emptyset$.

In the additive case, since

$$\mu(A_{(i)}) = \mu(\{(i)\}) + \mu(\{(i+1)\}) + \dots + \mu(\{(n)\}) = \mu(\{(i)\}) + \mu(A_{(i+1)}) \quad (16)$$

the Choquet integral reduces to a weighted mean,

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^n [\mu(A_{(i)}) - \mu(A_{(i+1)})] x_{(i)} = \sum_{i=1}^n \mu(\{(i)\}) x_{(i)} = \sum_{i=1}^n \mu(\{i\}) x_i \quad (17)$$

where the weights are given by $w_i = \mu(\{i\})$, for $i = 1, \dots, n$.

A capacity μ can be equivalently represented by its Möbius transform m_μ [42].

Definition 3 Let μ be a capacity on N . The Möbius transform associated with the capacity μ is defined as

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \quad T \subseteq N \quad (18)$$

where s and t denote the cardinality of the coalitions S and T , respectively.

Conversely, given the Möbius transform m_μ , the associated capacity μ is obtained as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \quad T \subseteq N. \quad (19)$$

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \quad \sum_{T \subseteq N} m_\mu(T) = 1 \quad (20)$$

and the monotonicity condition is expressed as follows [39, 11],

$$\sum_{S \subseteq T} m_\mu(S \cup \{i\}) \geq 0 \quad i = 1, \dots, n \quad T \subseteq N \setminus \{i\}. \quad (21)$$

This form of monotonicity condition derives from the original monotonicity condition in Definition 1, expressed as $\mu(T \cup \{i\}) - \mu(T) \geq 0$ for all $i \in N$ and $T \subseteq N \setminus \{i\}$.

The Choquet integral in Definition 2 can be expressed in terms of the Möbius transform in the following way [36],

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{T \subseteq N} m_\mu(T) \min_{i \in T} (x_i). \quad (22)$$

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [28] introduced the concept of k -additive capacities.

A capacity μ is said to be k -additive [28, 29] if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.

We consider now, in particular, the 1-additive (or simply additive) case and the 2-additive case, and we revisit formulas (19) - (22).

- In the additive case, the decomposition formula (19) takes the simple form

$$\mu(T) = \sum_{i \in T} m_\mu(\{i\}) \quad T \subseteq N, \quad (23)$$

the boundary conditions (20) reduce to

$$m_\mu(\emptyset) = 0 \quad \sum_{i \in N} m_\mu(\{i\}) = 1 \quad (24)$$

and the monotonicity condition (21) reduces to

$$m_\mu(\{i\}) \geq 0 \quad i = 1, \dots, n. \quad (25)$$

Moreover, for additive capacities, the Choquet integral in (22) reduces to

$$\mathcal{C}_\mu(x_1, \dots, x_n) = \sum_{i \in N} m_\mu(\{i\}) x_i. \quad (26)$$

- In the 2-additive case, the decomposition formula (19) takes the form

$$\mu(T) = \sum_{\{i\} \subseteq T} m_\mu(\{i\}) + \sum_{\{i, j\} \subseteq T} m_\mu(\{i, j\}) \quad T \subseteq N, \quad (27)$$

the boundary conditions (20) reduce to

$$m_\mu(\emptyset) = 0 \quad \sum_{\{i\} \subseteq N} m_\mu(\{i\}) + \sum_{\{i, j\} \subseteq N} m_\mu(\{i, j\}) = 1 \quad (28)$$

and the monotonicity condition (21) reduces to

$$m_\mu(\{i\}) \geq 0 \quad m_\mu(\{i\}) + \sum_{j \in T} m_\mu(\{i, j\}) \geq 0 \quad i = 1, \dots, n \quad T \subseteq N \setminus \{i\}. \quad (29)$$

Moreover, for 2-additive capacities, the Choquet integral in (22) reduces to

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{\{i\} \subseteq N} m_\mu(\{i\}) x_i + \sum_{\{i, j\} \subseteq N} m_\mu(\{i, j\}) \min(x_i, x_j). \quad (30)$$

4 Symmetric Capacities and Choquet Integrals

We examine the basic definitions and results presented in the previous section in the particular case of symmetric capacities and Choquet integrals.

Definition 4 *A capacity μ is said to be symmetric if it depends only on the cardinality of the coalition considered*

$$\mu(T) = \mu(t) \quad \text{where} \quad t = |T|. \quad (31)$$

Accordingly, for the Möbius transform m_μ associated with a symmetric capacity μ we use the notation

$$m_\mu(T) = m_\mu(t) \quad \text{where} \quad t = |T|. \quad (32)$$

Consider a Choquet integral with respect to a symmetric capacity μ . Then the Choquet integral reduces to an Ordered Weighted Averaging (OWA) function [50],

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i \in N} [\mu(n - i + 1) - \mu(n - i)] x_{(i)} = \sum_{i \in N} w_i x_{(i)} = A(\mathbf{x}) \quad (33)$$

where

$$w_i = \mu(n - i + 1) - \mu(n - i) \quad (34)$$

correspond to the OWA weights. The traditional form of OWA functions as introduced by Yager [50] (OWA operators) is as follows,

$$A(\mathbf{x}) = \sum_{i \in N} \tilde{w}_i x_{[i]} \quad (35)$$

where $\tilde{w}_i = w_{n-i+1}$ and $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$.

For a symmetric capacity μ , (24) - (25) and (28) - (29) take the following form.

- In the additive case the boundary conditions (24) reduce to

$$m_\mu(0) = 0 \quad n m_\mu(1) = 1 \quad (36)$$

and the monotonicity condition (25) reduces to

$$m_\mu(1) \geq 0. \quad (37)$$

From the boundary conditions (36) we have $m_\mu(1) \geq 1/n$ and the OWA function is simply the arithmetic mean.

- In the 2-additive case the boundary conditions (28) reduce to

$$m_\mu(0) = 0 \quad n m_\mu(1) + \frac{n(n-1)}{2} m_\mu(2) = 1 \quad (38)$$

and the monotonicity condition (29) reduces to

$$m_\mu(1) \geq 0 \quad m_\mu(1) + t m_\mu(2) \geq 0 \quad 1 \leq t \leq n-1. \quad (39)$$

In the next section we present a detailed treatment of the 2-additive symmetric case.

5 Symmetric Capacities and Choquet Integrals: 2-additive case

Consider now the 2-additive symmetric case as discussed in the previous section. Let

$$\alpha = m_\mu(1) \quad \beta = m_\mu(2). \quad (40)$$

From the boundary conditions (38) we have

$$n\alpha + \frac{n(n-1)}{2}\beta = 1 \quad \alpha = \frac{1}{n} - \frac{n-1}{2}\beta. \quad (41)$$

From the monotonicity condition (39) follows

$$\alpha \geq 0 \quad \alpha + (n-1)\beta \geq 0 \quad (42)$$

where the second constraint corresponds to the dominating worst case $t = n-1$ in (39). Substituting α as in (41) in the two conditions (42) we obtain

$$-\frac{2}{n(n-1)} \leq \beta \leq \frac{2}{n(n-1)}. \quad (43)$$

Consider now the OWA operator as in (33) and (34),

$$A(\mathbf{x}) = \sum_{i=1}^n w_i x_{(i)} \quad w_i = \mu(n-i+1) - \mu(n-i). \quad (44)$$

In the 2-additive case we obtain

$$w_i = \alpha + (n-i)\beta = \frac{1}{n} + \frac{n-2i+1}{2}\beta \quad (45)$$

where β is subject to the constraints (43).

Introducing the notation $u_i = (n-2i+1)/2$, $i = 1, \dots, n$, notice that $\sum_{i=1}^n u_i = 0$ and the coefficients u_i , $i = 1, \dots, n$ are linearly decreasing $u_1 > u_2 > \dots > u_n$ with $u_1 = (n-1)/2$ and $u_n = -(n-1)/2$.

The main result of the paper is then the following.

Proposition 1 *Any 2-additive OWA function can be written as*

$$A(\mathbf{x}) = \bar{x} - \frac{1}{2}\beta n^2 G_A^c(\mathbf{x}) \quad (46)$$

where β is a free parameter subject to the constraints $-\frac{2}{n(n-1)} \leq \beta \leq \frac{2}{n(n-1)}$.

The proof follows straightforwardly from (44) - (45), associated to the constraints (43), and the definition of the classical Gini inequality index (1).

Given that

$$A(\mathbf{x}) = \sum_{i=1}^n w_i x_{(i)} = \sum_{i=1}^n \frac{2 + \beta(n^2 - 2in + n)}{2n} x_{(i)} \quad (47)$$

we must have $\beta \geq 0$ in order to have non-increasing weights. The strict case $\beta > 0$ corresponds to the well-known Ben Porath and Gilboa's formula [3] for Weymark's generalized Gini welfare functions with linearly decreasing (inequality averse) weight distributions, see also [31].

In particular, with $\beta = 2/n^2$ we obtain the classical Gini welfare function

$$A(\mathbf{x}) = A_G^c(\mathbf{x}) \quad \alpha = \frac{1}{n^2} \quad \beta = \frac{2}{n^2}. \quad (48)$$

Regarding the choice of the parameter values α and β , we introduce

$$a = n\alpha \quad b = \frac{n(n-1)}{2}\beta \quad (49)$$

and then the boundary and monotonicity constraints (41) - (42) take the simple form

$$a + b = 1 \quad a \geq 0 \quad a + 2b \geq 0 \quad (50)$$

from which we obtain $a = 1 - b$ and $-1 \leq b \leq 1$. In this notation the classical Gini case $\alpha = 1/n^2$ and $\beta = 2/n^2$ corresponds to $a = 1/n$ and $b = (n-1)/n$.

Other interesting parameters choices for a , b could be $a = k/n$ and $b = (n-k)/n$ with $k = 0, \dots, n$. In the case $k = 0$ all the Choquet capacity structure lies in the interactions $\beta = m_\mu(2)$, the case $k = 1$ corresponds to the classical Gini inequality index, and in the remaining cases $k \geq 2$ increasingly weak structure is associated to interaction, towards the additive case $k = n$.

References

- [1] Aristondo, O., Garcia Lapresta, J. L., Lasso de la Vega, C., Marques Pereira, R. A., Urrutia, A. M.: The Gini index, the dual decomposition of aggregation functions, and the consistent measurement of inequality. *International Journal of Intelligent Systems* **27**, (2) 132–152 (2012)
- [2] Atkinson, A. B.: On the measurement of inequality. *Journal of Economic Theory* **2**, 244–263 (1970)
- [3] Ben Porath, E., Gilboa, I.: Linear measures, the Gini index, and the income-equality trade-off. *Journal of Economic Theory* **2**, (64) 443–467 (1994)
- [4] Bossert, W.: An axiomatization of the single-series Ginis. *Journal of Economic Theory* **50**, (1) 82–92 (1990)
- [5] Blackorby, C., Donaldson, D.: Measures of relative equality and their meaning in terms of social welfare. *Journal of Economic Theory* **18**, 59–80 (1978)
- [6] Blackorby, C., Donaldson, D.: A theoretical treatment of indices of absolute inequality. *International Economic Review* **21**, (1) 107–136 (1980)
- [7] Calvo, T., De Baets, B.: Aggregation operators defined by k -order additive/maxitive fuzzy measures. *International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems* **6**, (6) 533–550 (1998)
- [8] Cao-Van, K., De Baets, B.: A decomposition of k -additive Choquet and k -maxitive Sugeno integrals. *International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems* **2**, (9) 127–143 (2001)
- [9] Chakravarty, S.: Extended Gini indices of inequality. *International Economic Review* **29**, (1) 147–156 (1988)

- [10] Chateauneuf, A.: Modelling attitudes towards uncertainty and risk through the use of Choquet integral. *Annals of Operations Research* **52**, (1) 3–20 (1994)
- [11] Chateauneuf, A., Jaffray, J. Y.: Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences* **17**, (3) 263–283 (1989)
- [12] Choquet, G.: Theory of capacities. *Annales de l’Institut Fourier* **5**, 131–295 (1953)
- [13] Denneberg, D.: *Non-Additive Measure and Integral*. Kluwer Academic Publishers, Dordrecht (1994)
- [14] Donaldson, D., Weymark, J. A.: A single-parameter generalization of the Gini indices of inequality. *Journal of Economic Theory* **22**, (2) 67–86 (1980)
- [15] Dorfman, R.: A Formula for the Gini Coefficient. *The Review of Economics and Statistics* **61**, (1): 146–149 (1979)
- [16] Ebert, U.: Measurement of inequality: An attempt at unification and generalization. *Social Choice Welfare* **5**, (2) 147–169 (1988)
- [17] Fodor, J., Marichal, J. L., Roubens, M: Characterization of the ordered weighted averaging operators. *IEEE Trans. on Fuzzy Systems* **3**, (2) 236–240 (1995)
- [18] Gajdos, T.: Measuring inequalities without linearity in envy: Choquet integrals for symmetric capacities. *Journal of Economic Theory* **106**, (1) 190–200 (2002)
- [19] Garcia Lapresta, J. L., Marques Pereira, R. A.: The self-dual core and the anti-self-dual remainder of an aggregation operator. *Fuzzy Sets and Systems* **159**, (1) 47–62 (2008)
- [20] Garcia Lapresta, J. L., Lasso de la Vega, C., Marques Pereira, R. A., Urrutia, A. M.: A class of poverty measures induced by the dual decomposition of aggregation functions. *International Journal of Uncertainty, Fuzziness and Knowledge Based Systems* **18**, (4) 493–511 (2010)
- [21] Gastwirth, J. L.: The Estimation of the Lorenz Curve and Gini Index. *The Review of Economics and Statistics* **54**, (3) 306–316 (1972)
- [22] Gilboa, I., Schmeidler, D.: Additive representations of non-additive measures and the Choquet integral. *Annals of Operations Research* **52**, (1) 43–65 (1994)
- [23] Gilboa, I., Schmeidler, D.: Canonical representation of set functions. *Mathematics of Operations Research* **20**, (1) 197–212 (1995)
- [24] Gini, C.: *Variabilità e Mutabilità Contributo allo Studio delle Distribuzioni e delle Relazioni Statistiche*. C. Cuppini, Bologna (1912)
- [25] Gini, C.: Measurement of Inequality of Incomes. *The Economic Journal* **31**, (121) 124–126 (1921)

- [26] Grabisch, M.: Fuzzy integral in multicriteria decision making. *Fuzzy Sets and Systems* **69**, (3) 279–298 (1995)
- [27] Grabisch, M.: The application of fuzzy integrals in multicriteria decision making. *European Journal of Operational Research* **89**, (3) 445–456 (1996)
- [28] Grabisch, M.: k -order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems* **92**, (2) 167–189 (1997)
- [29] Grabisch, M.: Alternative representations of discrete fuzzy measures for decision making. *International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems* **5**, (5) 587–607 (1997)
- [30] Grabisch, M.: Alternative representations of OWA operators. In: Yager, R. R., Kacprzyk, J. (eds.) *The Ordered Weighted Averaging Operators: Theory and Applications*, pp. 73–85. Kluwer Academic Publishers, Dordrecht (1997)
- [31] Grabisch, M.: k -Additive measures: recent issues and challenges. In: *Proc. 5th International Conference on Soft Computing and Information/Intelligent Systems*, Izuka, Japan, pp. 394–397 (1998)
- [32] Grabisch, M., Labreuche, C.: Fuzzy measures and integrals in MCDA. In: Figueira, J., Greco, S., Ehrgott, M. (eds) *Multiple Criteria Decision Analysis*, pp. 563–604. Springer, Heidelberg (2005)
- [33] Grabisch, M., Labreuche, C.: A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *4OR* **6**, (1) 1–44 (2008)
- [34] Grabisch, M., Labreuche, C.: A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *Annals of Operations Research* **175**, (1) 247–286 (2010)
- [35] Grabisch, M., Kojadinovich, I., Meyer, P.: A review of methods for capacity identification in Choquet integral based multi-attribute utility theory: Applications of the Kappalab R package. *European Journal of Operational Research* **186**, (2) 766–785 (2008)
- [36] Marichal, J. L.: *Aggregation operators for multicriteria decision aid*. Ph.D. Thesis, University of Liège, Liège, Belgium (1998)
- [37] Mayag, B., Grabisch, M., Labreuche, C.: A representation of preferences by the Choquet integral with respect to a 2-additive capacity. *Theory and Decision* **71**, (3) 297–324 (2011)
- [38] Mayag, B., Grabisch, M., Labreuche, C.: A characterization of the 2-additive Choquet integral through cardinal information. *Fuzzy Sets and Systems* **184**, (1) 84–105 (2011)
- [39] Miranda, P., Grabisch, M.: Optimization issues for fuzzy measures. *International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems* **7**, (6) 545–560 (1999)
- [40] Miranda, P., Grabisch, M., Gil, P.: Axiomatic structure of k -additive capacities. *Mathematical Social Sciences* **49**, (2) 153–178 (2005)

- [41] Murofushi, T., Sugeno, M.: Some quantities represented by the Choquet integral. *Fuzzy Sets and Systems* **2**, (56) 229–235 (1993)
- [42] Rota, G. C.: On the foundations of combinatorial theory I. Theory of Möbius functions, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **2**, (4) 340–368 (1964)
- [43] Schmeidler, D.: Integral representation without additivity. *Proceedings of the American Mathematical Society* **97**, (2) 255–261 (1986)
- [44] Schmeidler, D.: Subjective probability and expected utility without additivity. *Econometrica* **57**, (3) 571–587 (1989)
- [45] Sen, A.: *On Economic Inequality*. Clarendon Press, Oxford (1973)
- [46] Sugeno, M.: *Theory of fuzzy integrals and its applications*. Ph.D. Thesis, Tokyo Institut of Technology, Japan (1974)
- [47] Weymark, J. A.: Generalized Gini inequality indices. *Mathematical Social Sciences* **1**, (4) 409–430 (1981)
- [48] Yaari, M.: The dual theory of choice under risk. *Econometrica* **55**, (1) 95–115 (1987)
- [49] Yaari, M.: A controversial proposal concerning inequality measurement. *Journal of Economic Theory* **44**, (2) 381–397 (1988)
- [50] Yager, R. R.: On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. on Systems, Man and Cybernetics* **18**, (1) 183–190 (1988)

