

**About arithmetic-geometric
multidistances**

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ABOUT ARITHMETIC-GEOMETRIC MULTIDISTANCES

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Abstract

In a previous paper (see [7]) we considered the family of multi-argument functions called multidistances, introduced in some recent papers (see [1]-[6]) by J.Martin and G.Mayor , which extend to n -dimensional ordered lists of elements the usual concept of distance between a couple of points in a metric space. In particular Martin and Mayor investigated three classes of multidistances, that is Fermat, sum-based and OWA- based multidistances. In this note we introduce a new family of multidistance functions, which are a generalization of the sum-based multidistances and we call them arithmetic- geometric multidistances.

Keywords: multidistance, sum-based multidistances, arithmetic- geometric multidistances

1. INTRODUCTION

In some recent papers (see [1] –[6]) J.Martin and G.Mayor (M. and M. in the sequel) proposed a formal definition of a multi-argument function distance.

In their papers, the formal definition of a distance function is extended to apply to collections of more than two elements. The measure proposed by the authors applies to n -dimensional ordered lists of elements. The authors give the definition of (weak) multidistance and strong multidistance and present significant examples of multidistance functions: in particular Fermat multidistances, sum-based multidistances and OWA-based multidistances.

In this note we introduce a new family of multidistance functions , which are a generalization of the sum-based multidistances and can be called arithmetic-geometric multidistances.

2. NOTATIONS AND DEFINITIONS

We recall briefly the formal definition of multidistance functions. (see, for example, [7], for further details). Given a set X , let \vec{X} be the collection of all n -dimensional lists of elements of X with $n=1,2,\dots$. In other words, we call \vec{X} the set given by $\bigcup_{n=1}^{\infty} X^n$.

The definition of multidistance function over the set X is the following:

DEFINITION OF MULTIDISTANCE

A function $D: \vec{X} \rightarrow [0, \infty]$ is a **multidistance** on a set X if the following properties hold, for all n and for all $x_1, \dots, x_n, y \in X$:

- (m1) $D(x_1, \dots, x_n) = 0$ if and only if $x_i = x_j$ for all $i, j = 1, \dots, n$
- (m2) $D(x_1, \dots, x_n) = D(x_{\pi(1)}, \dots, x_{\pi(n)})$ for any permutation π on $1, \dots, n$
- (m3) $D(x_1, \dots, x_n) \leq D(x_1, y) + \dots + D(x_n, y)$

REMARK

Note that if D is a multidistance on a set X , then the restriction of D to X^2 is an ordinary distance function on X .

In [2] M. and M. , starting from an ordinary distance function, introduce several kinds of multidistance functions. In particular, they consider Fermat, sum-based and OWA-based multidistances. In this note we focus our attention on the so-called sum-based multidistances.

DEFINITION OF SUM-BASED MULTIDISTANCE

Consider an ordinary distance function d on X and define a function $D_\lambda : \vec{X} \rightarrow [0, \infty)$ as follows:

$$\begin{cases} D_\lambda(x_1) = 0 \\ D_\lambda(x_1, \dots, x_n) = \lambda(n) \sum_{i < j} d(x_i, x_j) \text{ for } n \geq 2 \end{cases}$$

In [4] M. and M. proved that D_λ is a multidistance if and only if:

- (i) $\lambda(2) = 1$
- (ii) $0 < \lambda(n) \leq \frac{1}{n-1}$ for any $n > 2$

In this case we call D_λ a sum-based multidistance.

3. ARITHMETIC-GEOMETRIC MULTIDISTANCES

Now we introduce a new family of multidistance functions and we call them **arithmetic-geometric multidistances**. The definition is as follows. Consider an ordinary distance function d on X and define a function $D_\lambda^{AG} : \vec{X} \rightarrow [0, \infty)$ in the following way:

$$\begin{cases} D_\lambda^{AG}(x_1) = 0 \\ D_\lambda^{AG}(x_1, \dots, x_n) = \lambda(n) \sum_{i < j} d(x_i, x_j) + \left[\frac{n}{2} - \binom{n}{2} \lambda(n) \right] \left(\prod_{i < j} d(x_i, x_j) \right)^{\frac{1}{\binom{n}{2}}} \end{cases}$$

where $0 < \lambda(n) \leq \frac{1}{n-1}$ for $n \geq 2$

REMARK 1

Note that $\frac{n}{2} - \binom{n}{2} \lambda(n) = \frac{n}{2} [1 - (n-1)\lambda(n)] \geq 0$ by definition.

REMARK 2

Note that $D_\lambda^{AG}(x_1, x_2) = \lambda(n)d(x_1, x_2) + [1 - \lambda(n)]d(x_1, x_2) = d(x_1, x_2)$

That is, the restriction of D_λ^{AG} on X^2 is simply d .

We can now prove the following

THEOREM

D_λ^{AG} is a multidistance function.

Proof.

It is easy to verify that conditions (m1) and (m2) are satisfied. Then we have to prove condition (m3), that is

$$D_\lambda^{AG}(x_1, \dots, x_n) \leq D_\lambda^{AG}(x_1, y) + \dots + D_\lambda^{AG}(x_n, y) \quad \forall y \in X$$

We start from the arithmetic-geometric inequality

$$\left(\prod_{i < j} d(x_i, x_j) \right)^{\frac{1}{\binom{n}{2}}} \leq \frac{1}{\binom{n}{2}} \sum_{i < j} d(x_i, x_j) = \frac{2}{n(n-1)} \sum_{i < j} d(x_i, x_j)$$

On the other hand, we can observe that

$$\sum_{i < j} d(x_i, x_j) \leq \sum_{i < j} [d(x_i, y) + d(x_j, y)] = (n-1) \sum_{i=1}^n d(x_i, y) \quad \text{that is}$$

$\frac{1}{n-1} \sum_{i < j} d(x_i, x_j) \leq \sum_{i=1}^n d(x_i, y)$. Thanks to this inequality then we get

$$\left(\prod_{i < j} d(x_i, x_j) \right)^{\frac{1}{\binom{n}{2}}} \leq \frac{2}{n} \sum_{i=1}^n d(x_i, y)$$

By using again the same inequality we can write

$$\begin{aligned}
D_{\lambda}^{AG}(x_1, \dots, x_n) &= \lambda(n) \sum_{i < j} d(x_i, x_j) + \left[\frac{n}{2} - \binom{n}{2} \lambda(n) \right] \left(\prod_{i < j} d(x_i, x_j) \right)^{\frac{1}{\binom{n}{2}}} \leq \\
&\leq \lambda(n)(n-1) \sum_{i=1}^n d(x_i, y) + \left[\frac{n}{2} - \binom{n}{2} \lambda(n) \right] \frac{2}{n} \sum_{i=1}^n d(x_i, y) = \\
&= \lambda(n)(n-1) \sum_{i=1}^n d(x_i, y) + [1 - (n-1)\lambda(n)] \sum_{i=1}^n d(x_i, y) = \sum_{i=1}^n d(x_i, y)
\end{aligned}$$

We conclude that $D_{\lambda}^{AG}(x_1, \dots, x_n) \leq \sum_{i=1}^n d(x_i, y)$, that is

$$D_{\lambda}^{AG}(x_1, \dots, x_n) \leq D_{\lambda}^{AG}(x_1, y) + \dots + D_{\lambda}^{AG}(x_n, y) \quad \forall y \in X. \text{ And the proof is complete. } \square$$

Last thing, we can easily establish a simple relationship between ordinary sum-based multidistances D_{λ} and our arithmetic-geometric multidistances D_{λ}^{AG} .

Note that, if we set $\lambda^*(n) = \frac{1}{n-1}$, we have $\frac{n}{2} - \binom{n}{2} \lambda^*(n) = 0$ and then

$$D_{\lambda^*}^{AG}(x_1, \dots, x_n) = \frac{1}{n-1} \sum_{i < j} d(x_i, x_j) = D_{\lambda^*}(x_1, \dots, x_n)$$

Note also that, since $\frac{n}{2} - \binom{n}{2} \lambda(n) \geq 0$, we have obviously $D_{\lambda} \leq D_{\lambda}^{AG}$.

In general, thanks to the arithmetic-geometric inequality, we can write

$$\begin{aligned}
D_{\lambda}^{AG}(x_1, \dots, x_n) &= \lambda(n) \sum_{i < j} d(x_i, x_j) + \left[\frac{n}{2} - \binom{n}{2} \lambda(n) \right] \left(\prod_{i < j} d(x_i, x_j) \right)^{\frac{1}{\binom{n}{2}}} \leq \\
&\leq \lambda(n) \sum_{i < j} d(x_i, x_j) + \left[\frac{n}{2} - \binom{n}{2} \lambda(n) \right] \frac{1}{\binom{n}{2}} \sum_{i < j} d(x_i, x_j) = \\
&= \left\{ \lambda(n) + \left[\frac{1}{n-1} - \lambda(n) \right] \right\} \sum_{i < j} d(x_i, x_j) = \frac{1}{n-1} \sum_{i < j} d(x_i, x_j) = D_{\lambda^*}(x_1, \dots, x_n)
\end{aligned}$$

To sum up, we can conclude that

$$D_{\lambda}(x_1, \dots, x_n) \leq D_{\lambda}^{AG}(x_1, \dots, x_n) \leq D_{\lambda^*}^{AG}(x_1, \dots, x_n) = D_{\lambda^*}(x_1, \dots, x_n).$$

6. CONCLUSIONS

We consider a special class of multi-argument functions, called multidistances, introduced in some recent papers (see [1] –[6]) by J.Martin and G.Mayor , which extend to n -dimensional ordered lists of elements the usual concept of distance between a couple of points in a metric space. They considered several kinds of multidistances and in particular the family of the so-called sum-based multidistances. In this note we introduce a new family of multidistance functions , which are a generalization of the sum-based multidistances, and we call them arithmetic-geometric multidistances.

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