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A regular multidistance among fuzzy numbers

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A REGULAR MULTIDISTANCE AMONG FUZZY NUMBERS

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Abstract

We consider the family of multi-argument functions called multidistances, which capture the idea of distance for collections of more than two points in a metric space. Usually multidistances are defined among crisp objects. In this note we propose a multidistance defined among fuzzy numbers which satisfies the following conditions: it must take into account all distances between all pairs of fuzzy data, must be regular and easy to compute.

Keywords: multidistance, regularity, fuzzy numbers

1. INTRODUCTION

In some recent papers [3, 4, 5, 6, 7 and 8] Martin and Mayor proposed a formal definition of a multi-argument distance function, called multidistance function, which captures the idea of distance for collections of more than two points in a metric space. In [7] the authors state that , in addition to their intrinsic mathematical interest, multidistances have many potential applications: distance-based clustering, pattern recognition and so on. In their works they always consider multidistances defined among crisp data. Since we usually deal with data which are imprecise by their very nature, we think that it is worth to consider also multidistance functions among fuzzy data. In this note we propose a multidistance defined among fuzzy numbers which satisfies the following conditions: it must take into account all distances between all pairs of fuzzy data, must be regular and easy to compute. In section 2 definitions and notations about fuzzy numbers are presented, and in section 3 we prove that such a multidistance really satisfies the above conditions.

2. DEFINITIONS AND NOTATIONS

Since there are a number of ways of defining fuzzy numbers (see, for example, [1]and [2]), we point out that in this paper the following definition of fuzzy number is considered:

DEFINITION OF FUZZY NUMBER

The fuzzy set A of the real line , with membership function $\mu_A(\cdot)$, is a fuzzy number iff:

1) The α-level set A^α= $\{x \in R | \mu_A(x) \geq \alpha\}$, for all $\alpha \in [0,1]$, is a convex set.
2) $\mu_A(\cdot)$ is an upper semi continuous function.
3) $A \; is \; normalized$, i.e. there exists $m \in R$ such that $\mu_A(m) = 1$.
4) The support of A , $supp(A) = \{x \in R | \mu_A(x) \; > 0\}$, is a bounded set of R.

From the previous definition it follows that the level sets of a fuzzy number A are closed and bounded real intervals : A^α=$\{x \in R | a_1(\alpha) \leq x \leq a_2(\alpha)\} = [a_1(\alpha), a_2(\alpha)]$ for all $\alpha \in (0, 1]$. When $\alpha = 0$, we define A^0 as the closure of $supp(A)$. It is easy to verify that

1) $a_1(\cdot)$ is a bounded increasing function
2) $a_2(\cdot)$ is a bounded decreasing function
3) $a_1(1) \leq a_2(1)$

The set A^1=$[a_1(1), a_2(1)]$ is called the $mode$ of A. When $a_1(1) = a_2(1) = m$, that is $A^1 = \{m\}$, we say that A is a unimodal fuzzy set. In the sequel we will denote by $\mathcal{F}$ the set of fuzzy numbers.

REMARK

A closed and bounded interval $I = [a, b]$ can be considered a fuzzy number whose membership function is equal to the characteristic function of I, that is

$$\mu_I(x) = \begin{cases} 1 \; if \; x \in [0,1] \\ 0 \; if \; x \notin [0,1] \end{cases}$$

In this case $a_1(\alpha) = a$ and $a_2(\alpha) = b$ for all $\alpha \in [0,1]$. If $a = b = m$ then the interval I becomes a real number. It follows that a multidistance among fuzzy numbers is a generalization of a multidistance among closed and bounded intervals which is a generalization of a multidistance among real numbers.

DEFINITION OF MULTIDISTANCE

We recall briefly the formal definition of a multidistance function. Given a set X, let $\overrightarrow{X}$ be the collection of all n-dimensional lists of elements of X with $n = 1,2, \dots$ In other words, we call $\overrightarrow{X}$ the set given by $\bigcup_{n=1}^{\infty} X^n$. The definition of multidistance function over the set X is the following:

A function $D: \overrightarrow{X} \rightarrow [0, \infty)$ is a (weak) **multidistance** on a set X if the following properties hold, for all n and for all $x_1, x_2, \dots, x_n, y \in X$:

$(m1)\ D(x_1, \dots, x_n) = 0 \; if \; and \; only \; if \; x_i{=}x_j \;\; for \; all \; i, j = 1, \dots, n$

$(m2)\ D(x_1, \dots, x_n) = \; D\big(x_{\pi(1)}, \dots, x_{\pi(n)}\big) \; for \; any \; permutation \; \pi \; on \; 1, \dots, n$

$(m3)\ D(x_1, \dots, x_n) \leq D(x_1, y) + \cdots + D(x_n, y)$

We say that D is a **strong multidistance** if it fulfils $(m1), (m2)$ and

$(m3')\ D(\vec{x}_1, \dots, \vec{x}_k) \leq D(\vec{x}_1, \vec{y}) + \cdots + D(\vec{x}_k, \vec{y}) \; for \; all \; \vec{x}_1, \dots, \vec{x}_k \, , \vec{y} \in \overrightarrow{X}$

A very interesting property of multidistances, namely their regularity, is defined as follows. A multidistance $D: \overrightarrow{X} \rightarrow [0, \infty)$ is said to be **regular** iff:

$D(\vec{x}, y) \geq D(\vec{x}) \; for \; all \; \vec{x} \in \overrightarrow{X}, y \in X.$

That is, the multidistance of a list cannot decrease when a new element is added to the list. Note that a strong multidistance is, by definition, a regular multidistance. Then regular multidistances can be considered in-between the classes of weak and strong multidistances. Regularity is a very desirable property, in particular when using multidistances applied to clustering, pattern recognition or similar topics.

REMARK

Note that if D is a multidistance on a set X, then the restriction of D to X^2 is an ordinary distance function on X .

3. A REGULAR MULTIDISTANCE AMONG FUZZY NUMBERS

Our goal is to build up a multidistance among fuzzy numbers which has three properties: it must take into account all distances between all pairs of fuzzy data, must be regular and easy to compute. We built up an ordinary distance function between fuzzy numbers and then investigated three major types of multidistances proposed by Martin and Mayor (see[8]), namely Fermat, sum-based and OWA -based multidistances. We discarded Fermat multidistance, though it is a regular multidistance, because in this case the calculations are too costly (this multidistance requires the calculation of the minimum of the distances over the whole set of fuzzy numbers). We eliminated OWA- based multidistances because in general they are not regular multidistances (see[6]). Then we came to the conclusion that (within these three classes) the only multidistance function satisfying the above conditions is a multidistance of the sum-based type with a multiplying factor that makes it regular. Our proposal in formal terms is as follows.

Given the set $\mathcal{F}$ of fuzzy numbers, let $\vec{\mathcal{F}}$ be the collection of all n-dimensional lists of fuzzy numbers with $n = 1,2,\ldots$. In other words, $\vec{\mathcal{F}}$ is the set given by $\bigcup_{n=1}^{\infty} \mathcal{F}^n$. In the sequel we'll consider a generic list of fuzzy numbers $A_1, \ldots, A_n$. Please remember that , by definition, every fuzzy number A_i of the list is fully identified by its level sets $A_i^\alpha = \left[a_1^i(\alpha), a_2^i(\alpha)\right]$ for $\alpha \in [0,1]$ and $i = 1,2, \ldots, n$.

DEFINITION OF A REGULAR MULTIDISTANCE AMONG FUZZY NUMBERS

Let us define a function $D^{\mathcal{F}}: \vec{\mathcal{F}} \to [0, \infty)$ as follows:

$$D^{\mathcal{F}}(A_1, \ldots, A_n) = \frac{1}{n-1} \sum_{1 \le i < j \le n} \sup_{0 \le \alpha \le 1} max\left\{\left|a_1^i(\alpha) - a_1^j(\alpha)\right|, \left|a_2^i(\alpha) - a_2^j(\alpha)\right|\right\} \qquad (1)$$

$where\; n \geq 2.$

We prove the following theorem.

THEOREM The function $D^{\mathcal{F}}$ is a regular multidistance on $\mathcal{F}$.

Proof of (m1). It is clear that $D^{\mathcal{F}}(A_1, \ldots, A_n) = 0$ iff

$max\left\{\left|a_1^i(\alpha) - a_1^j(\alpha)\right|, \left|a_2^i(\alpha) - a_2^j(\alpha)\right|\right\} = 0 \; for\; all\; \alpha \in [0,1]$,

that is iff $\left|a_1^i(\alpha) - a_1^j(\alpha)\right| = 0 \; and\; \left|a_2^i(\alpha) - a_2^j(\alpha)\right| = 0 \; for\; all\; \alpha \in [0,1]$,

that is iff $a_1^i(\alpha) = a_1^j(\alpha) \; and\; a_2^i(\alpha) = a_2^j(\alpha) \; for\; all\; \alpha \in [0,1]$.

But this means that $A_i = A_j \; for\; all\; i,j = 1,2, \ldots, n$. ■

Proof of (m2). It is obvious. ■

Proof of (m3). We have to prove that $D^{\mathcal{F}}(A_1, \ldots, A_n) \le D^{\mathcal{F}}(A_1, B) + \cdots + D^{\mathcal{F}}(A_n, B)$ for all $A_1, \ldots, A_n, B \in \mathcal{F}$. Consider the α –level set of B given by $B^\alpha = [b_1(\alpha), b_2(\alpha)]$ and note that, by definition (1), we get

$$D^{\mathcal{F}}(A_k, B) = \sup_{0 \le \alpha \le 1} max\left\{\left|a_1^k(\alpha) - b_1(\alpha)\right|, \left|a_2^k(\alpha) - b_2(\alpha)\right|\right\}$$

Then we must show that

$$\frac{1}{n-1}\sum_{1\le i<j\le n}\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}$$
$$\le\sum_{k=1}^{n}\sup_{0\le\alpha\le1} max\{|a_1^k(\alpha)-b_1(\alpha)|,|a_2^k(\alpha)-b_2(\alpha)|\}$$

Now, since $|a_1^i(\alpha)-a_1^j(\alpha)|\le|a_1^i(\alpha)-b_1(\alpha)|+|a_1^j(\alpha)-b_1(\alpha)|$ and

$|a_2^i(\alpha)-a_2^j(\alpha)|\le|a_2^i(\alpha)-b_2(\alpha)|+|a_2^j(\alpha)-b_2(\alpha)|$, we get

$max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}\le max\{|a_1^i(\alpha)-b_1(\alpha)|,|a_2^i(\alpha)-b_2(\alpha)|\}+$
$max\{|a_1^j(\alpha)-b_1(\alpha)|,|a_2^j(\alpha)-b_2(\alpha)|\}$

and then , a fortiori, we have

$$\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}\le\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-b_1(\alpha)|,|a_2^i(\alpha)-b_2(\alpha)|\}$$
$$+\sup_{0\le\alpha\le1} max\{|a_1^j(\alpha)-b_1(\alpha)|,|a_2^j(\alpha)-b_2(\alpha)|\}$$

It follows that

$$\frac{1}{n-1}\sum_{1\le i<j\le n}\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}\le$$

$$\frac{1}{n-1}\sum_{1\le i<j\le n}\Big[\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-b_1(\alpha)|,|a_2^i(\alpha)-b_2(\alpha)|\}$$
$$+\sup_{0\le\alpha\le1} max\{|a_1^j(\alpha)-b_1(\alpha)|,|a_2^j(\alpha)-b_2(\alpha)|\}\Big]$$

But we note that

$$\sum_{1\le i<j\le n}\Big[\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-b_1(\alpha)|,|a_2^i(\alpha)-b_2(\alpha)|\}$$
$$+\sup_{0\le\alpha\le1} max\{|a_1^j(\alpha)-b_1(\alpha)|,|a_2^j(\alpha)-b_2(\alpha)|\}\Big]$$
$$=(n-1)\sum_{k=1}^{n}\sup_{0\le\alpha\le1} max\{|a_1^k(\alpha)-b_1(\alpha)|,|a_2^k(\alpha)-b_2(\alpha)|\}$$

That is

$$\frac{1}{n-1}\sum_{1\le i<j\le n}\Big[\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-b_1(\alpha)|,|a_2^i(\alpha)-b_2(\alpha)|\}$$

$$+\sup_{0\le\alpha\le1} max\{|a_1^j(\alpha)-b_1(\alpha)|,|a_2^j(\alpha)-b_2(\alpha)|\}\Big]$$

$$=\sum_{k=1}^{n}\sup_{0\le\alpha\le1} max\{|a_1^k(\alpha)-b_1(\alpha)|,|a_2^k(\alpha)-b_2(\alpha)|\}$$

We conclude that

$$D^{\mathcal{F}}(A_1,\dots,A_n)=\frac{1}{n-1}\sum_{1\le i<j\le n}\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}\le$$
$$\sum_{k=1}^{n}\sup_{0\le\alpha\le1} max\{|a_1^k(\alpha)-b_1(\alpha)|,|a_2^k(\alpha)-b_2(\alpha)|\}=\sum_{k=1}^{n}D^{\mathcal{F}}(A_k,B) \qquad (2)$$

And the proof is complete. ■

Proof of regularity. We can easily verify that $D^{\mathcal{F}}$ is a regular multidistance, namely that

$$D^{\mathcal{F}}(A_1,\dots,A_n,B)\ge D^{\mathcal{F}}(A_1,\dots,A_n)\ for\ all\ A_1,\dots,A_n,B\in\mathcal{F}.$$

In fact, by definition (1), we get

$$D^{\mathcal{F}}(A_1,\dots,A_n,B)=\frac{1}{n}\Big[\sum_{1\le i<j\le n}\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}+$$
*k=1n*sup *0≤α≤1maxa1kα−b1α,a2kα−b2α*

and by using inequality (2) we can write

$$D^{\mathcal{F}}(A_1,\dots,A_n,B)$$
$$\ge\frac{1}{n}\sum_{1\le i<j\le n}\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}$$
$$+\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}=$$

$$\frac{1}{n-1}\sum_{1\le i<j\le n}\sup_{0\le\alpha\le1} max\{|a_1^i(\alpha)-a_1^j(\alpha)|,|a_2^i(\alpha)-a_2^j(\alpha)|\}= D^{\mathcal{F}}(A_1,\dots,A_n).\ ■$$

REMARK

Please note that if there is no fuzzyness, i.e. if all the data are closed and bounded intervals of the real line given by $A_1=[a_1^1,a_2^1]$,..., $A_n=[a_1^n,a_2^n]$,

then the formula reduces to

$$D^{\mathcal{F}}(A_1,\dots,A_n)=\frac{1}{n-1}\sum_{1\le i<j\le n} max\{|a_1^i-a_1^j|,|a_2^i-a_2^j|\}\ .$$

In the simplest case where the data are just real numbers given by $A_1=\{a^1\}$,..., $A_n=\{a^n\}$, the formula comes down to

$$D^{\mathcal{F}}(A_1, \dots, A_n) = \frac{1}{n-1} \sum_{1 \le i < j \le n} \left| a^i - a^j \right| ,$$

which is the only regular multidistance of the sum-based type generated by the ordinary Euclidean distance on the real line (see [6] or [9]).

4. CONCLUSIONS

In this note a multidistance among fuzzy numbers is proposed. This multidistance satisfies the following conditions: it takes into account all distances between all pairs of fuzzy data, it is regular and easy to compute. In a future work we intend to explore possible applications of this multidistance.

REFERENCES

[1] D. Dubois and H. Prade, Fuzzy numbers. An overview, in: JC Bezdek, Ed., The Analysis of Fuzzy Information, Vol. 1: Mathematics and Logic (CRC Press, Boca Raton, FL, 1987) (Also in: Tech. Rep. No. 219, LSI, Univ. P. Sabatier, Toulouse).

[2] R. Goetschel and W. Voxman, Topological properties of fuzzy numbers, Fuzzy Sets and Systems 9, 1983, pp. 87-99.

[3] J. Martin, G. Mayor, Aggregating pairwise distance values, in: Proceedings of the EUROFUSE 2009, Pamplona, Spain, 2009, pp.147-152

[4] J. Martin, G. Mayor, An axiomatic approach to T-multi-indistinguishabilities, in: Proceedings of the IFSA-EUSFLAT 2009, Lisbon, Portugal, 2009, pp.1723-1728

[5] J. Martin, G. Mayor, How separated Palma, Inca and Manacor are?, in: Proceedings of the International Summer School on Aggregation Operators, AGOP 2009, Palma, Spain, pp. 195-200.

[6] J. Martin, G. Mayor, Regular multidistances, in: XV Congreso Espagnol sobre Tecnologias y Logica Fuzzy, ESTYLF 2010, Huelva, 2010, pp. 297-301.

[7] J. Martin, G. Mayor, Some properties of multi-argument distances and Fermat multidistance , in: IPMU 2010, 2010, pp. 703-711.

[8] J. Martin, G. Mayor, Multi-argument distances, Fuzzy Sets and Systems 167, 2011, pp. 92-100.

[9] F. Molinari, About a new family of multidistances, Fuzzy Sets and Systems 195, 2012, pp.118-122.