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Simulating copula-based distributions and estimating tail probabilities by means of Adaptive Importance Sampling

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Abstract

Copulas are an essential tool for the construction of non-standard multivariate probability distributions. In the actuarial and financial field they are particularly important because of their relationship with non-linear dependence and multivariate extreme value theory. In this paper we use a recently proposed generalization of Importance Sampling, called Adaptive Importance Sampling, for simulating copula-based distributions and computing tail probabilities. Unlike existing methods for copula simulation, this algorithm is general, in the sense that it can be used for any copula whose margins have an unbounded support. After working out the details for Archimedean copulas, we develop an extension, based on an appropriate transformation of the marginal distributions, for sampling extreme value copulas. Extensive Monte Carlo experiments show that the method works well and its implementation is simple. An example with equity data illustrates the potential of the algorithm for practical applications.

Keywords: Adaptive Importance Sampling, Copula, Multivariate Extreme Value Theory, Tail probability.

JEL Classification: C15, C63.

1. Introduction

The dependence structure of financial assets is well-known to be an extremely important issue in non-life insurance and quantitative finance, both for risk management and for pricing. In the pioneering works about portfolio theory (Markowitz, 1952) and capital markets (Capital Asset Pricing Model: Sharpe, 1964; Lintner, 1965; Mossin, 1966), the multivariate normal distribution has played a key role. Although it is still a convenient model in standard cases, in recent years, with the increased degree of sophistication of markets and securities, its inadequacy as a general model has become clear. Difficulties are mostly related to either fat tails in the marginal distributions or non-linearity of the dependence structure.

Examples of these problems can be found in different areas. The standard approach to market risk measurement is based on the assumption of joint normality of risk factors, so that it mainly consists in estimating large covariance matrices. However, according to some well-known stylized facts concerning financial time series, such as jumps, heavy tails and volatility clustering (see Cont, 2001, for a review), the normality assumption is seldom justified, and non-standard distributions have to be used. Commonly adopted methodologies include Extreme Value Theory (EVT), GARCH and stochastic volatility. In the univariate case, estimating the parameters of the resulting distributions is usually feasible, but the extension to a multivariate setup is difficult both theoretically and computationally. Given that the complications are mostly related to the large number of parameters, it may be convenient to proceed in two steps. First, choosing a univariate model for the marginal distributions and estimating its parameters; second, studying the dependence structure and estimating the joint distribution. The latter task is typically accomplished by means of copulas. In particular, Extreme Value (EV) copulas are important when one is interested in the joint probability of rare events, as is frequently the case in insurance.

In credit risk management, the modelling of joint default probabilities is a crucial topic: in this field the distributions are "far from standard", and for the important class of threshold models the multivariate distributions are often obtained (explicitly or implicitly) by means of copulas; see McNeil *et al.* (2005, sect. 8.3). Furthermore, copulas are widely employed for pricing basket credit derivatives and CDO structures; see McNeil *et al.* (2005, sect. 9.7) or Bluhm

et al. (2002, chap. 7).

Copulas are a powerful tool for the construction of multivariate probability distributions. Although the main result concerning copulas dates back to more than fifty years ago (Sklar, 1959), recently there has been a spectacular advance both in theory and applications, especially in the financial industry. Given p marginal distribution functions $F_1, \ldots, F_p$, copulas allow to construct a multivariate distribution with a given dependence structure and marginals $F_1, \ldots, F_p$. Thus, this procedure of building a multidimensional distribution separates the marginal distributions and the dependence structure. The advantage is twofold: first, it guarantees a high degree of flexibility in the model building process; second, it allows to estimate separately the parameters of the marginal distributions and of the copula. From the computational point of view, this is usually a significant advantage: $p + 1$ "simple" optimizations are in general more feasible than a single "complicated" optimization.

Sampling copulas is an issue of primary importance, because analytical results are seldom available for copula-based distributions. The existing results can be grouped in three categories. As for *elliptical* copulas, simulation is easy, as it is based on the simulation of the multivariate Gaussian or Student-t distribution. In the case of *Archimedean* copulas, specific algorithms have been developed for the so-called LT-Archimedean copulas (McNeil *et al.*, 2005, p. 224). However, different algorithms are necessary for each copula. Moreover, they are sometimes involved, requiring the simulation of logarithmic series and/or stable random variables: see McNeil *et al.* (2005), Whelan (2004), Wu *et al.* (2006) and Hofert (2008). When the LT does not exist in closed form, and for non-Archimedean copulas, it is possible to use the inverse of the conditional distribution function of a variable given the remaining ones. However, this procedure is complicated and computationally cumbersome, because it requires high-order derivatives and a root-finding procedure.

In this paper we propose to use Adaptive Importance Sampling (IS) to generate an approximate sample from a given copula. The procedure only requires that the density of the copula be known in closed form. Moreover, it is general, because the algorithm is the same for any copula-based distribution, and can be employed for the simulation of non-Archimedean copulas, which are difficult to sample by means of other methods. Finally, it works well for any dimension.

Adaptive IS was first introduced by Cappé *et al.* (2004). Cappé *et al.* (2008)

extend it to general mixture classes, and Wraith *et al.* (2009) propose an interesting application. The method is an extension of standard IS because, at each iteration, a sample is simulated from the instrumental density and used to improve the IS density itself. So, 'adaptivity' means that the instrumental density sequentially provides a progressively better approximation of the target density. The method is also appealing because it rigorously formalizes the well-known fact that mixtures can be a very good approximation of most distributions.

The paper is organized as follows. Section 2 introduces the statistical methods used in the paper, namely standard Importance Sampling and Adaptive Importance Sampling. Section 3 reviews copulas and existing sampling algorithms. Section 4 develops Adaptive IS in the setup of copulas. Section 5 presents the results of the simulation experiments as well as a real-data application. Finally, Section 6 concludes.

2. Methodology

2.1. Preliminaries: Importance Sampling

Importance Sampling (Hammersley and Handscomb, 1964; see Casella and Robert, 2004, Sect. 3.3 or Glasserman, 2003, Sect. 4.6 for reviews) is a variance reduction technique. Unlike crude Monte Carlo (MC), IS does not simulate observations from the distribution of interest f, but from an instrumental distribution g that assigns "more weight" to the event of interest. The features of g are crucial, as they can result in either large efficiency gains or poor quality estimators. IS is one of the most effective variance reduction techniques, but there is no readily available procedure for finding the optimal instrumental density. Some tools, mainly based on tilted densities (Ross, 2006), are available when working with light-tailed distributions. For heavy-tailed distributions, the problem has to be solved differently, mostly on a case-by-case basis.

In order to formalize the setup, consider that most problems in statistics consist in evaluating the integral:

$$\mu = E_f(h(X)) = \int_{\mathcal{X}} h(x) f(x) dx, \tag{1}$$

where h is a real-valued function. If g is another density and $\text{supp}(g) \supset \text{supp}(f)$, (1) has the following alternative representation:

$$\mu = E_f(h(X)) = \int_{\mathcal{X}} h(x) \frac{f(x)}{g(x)} g(x) dx. \tag{2}$$

According to (2), instead of simulating from f as in standard MC, one can use the following algorithm.

Algorithm 1. *(Importance Sampling)*

- *Simulate* $x_1, \ldots, x_N$ *independently from* g*;*
- *Compute*

$$\hat{\mu}_{IS} = \frac{1}{N} \sum_{i=1}^{N} h(x_i) \frac{f(x_i)}{g(x_i)}. \tag{3}$$

The estimator $\hat{\mu}_{IS}$ is called an *importance sampling estimator* of μ, and g is the *importance sampling density*. The ratio $w(x) = f(x)/g(x)$ can be interpreted as a weight, so that (3) is a weighted mean. Unbiasedness, consistency and asymptotic normality of the estimator $\hat{\mu}_{IS}$ follow from the asymptotic theory of standard MC, under relatively mild regularity conditions (Casella and Robert, 2004, sect. 3.3).

Unlike the classical IS approach described so far, where g is chosen in advance and remains fixed, Adaptive IS updates the instrumental density as the simulation goes on. Thus, even though the initial instrumental density is not optimal, the algorithm eventually provides the 'best' (in a sense to be made more precise later) IS density.

2.2. Adaptive Importance Sampling

Adaptive Importance Sampling (or Population Monte Carlo) is a sampling algorithm first proposed in a Bayesian setup by Cappé *et al.* (2004). The basic idea is that, instead of using a fixed IS density g, one can use a sequence of importance densities $g^{(t)}$ $(t = 1, \ldots, T)$, aimed at approximating the target density f. The general formulation of the algorithm is as follows:

1. Simulate a first sample $\boldsymbol{x}_n = (x_1, \ldots, x_n)' \sim g^{(1)}$ by means of standard IS; compute the IS weights $w_j^{(1)} = f(x_j^{(1)})/g^{(1)}(x_j^{(1)})$ $(j = 1, \ldots, n)$.
2. Use this sample to approximate the moments of f and construct the updated importance function $g^{(2)}$.
3. Measure the goodness of the approximation by means of the relative entropy (or Kullback-Leibler divergence; see Rubinstein and Kroese, 2004, sect. 3.3) from the target:

$$K(f||g^{(2)}) = \int \log \left(\frac{f(x)}{g^{(2)}(x)} \right) f(x) dx. \tag{4}$$

4. 'Adjust' the density $g^{(2)}$ so that $K(f||g^{(2)}) \leq K(f||g^{(1)})$. More formally, compute $\min_{\boldsymbol{\theta}} K(f||g^{(2)})$, where $\boldsymbol{\theta}$ is the vector of parameters of $g^{(2)}$.
5. Repeat the preceding steps until some convergence criterion is met.

The first important issue is the choice of the functional form of g: it should be flexible enough to allow for a close match of the target and such that the minimization of (4) is feasible. According to Cappé *et al.* (2004, 2008), a convenient choice is a finite mixture of Normal distributions, namely

$$g^{(t)}(x) = g(x; \boldsymbol{\pi}^{(t)}, \boldsymbol{\theta}^{(t)}) = \sum_{d=1}^{D} \pi_d^{(t)} \phi_d(x; \boldsymbol{\theta}_d^{(t)}), \tag{5}$$

where $\boldsymbol{\pi}^{(t)} = (\pi_1^{(t)}, \ldots, \pi_D^{(t)})'$ is the vector of weights of the D mixture components and $\boldsymbol{\theta}^{(t)}$ is a $D \times 2$ matrix whose i-th row contains the parameters of the i-th normal density, i.e. $\boldsymbol{\theta}_i^{(t)} = (\mu_i^{(t)}, \sigma_i^{(t)2})$. If the target density is p-variate ($p \geq 2$), (5) is a p-variate Normal mixture and the dimensions of $\boldsymbol{\theta}$ change accordingly.

Notice that all parameters of (5) are *adaptable*, that is, they are supposed to be updated at each iteration of the algorithm. There is no reason that keeps us from using mixtures of different densities: Cappé *et al.* (2004, 2008) actually work out the details of the algorithm also for a mixture of Student-t densities, which are supposed to be a better option if the target is heavy-tailed.

According to these remarks, we are now in a position to give a more detailed description of the algorithm.

Algorithm 2. *(Adaptive Importance Sampling)*

1. *For $t = 1$:*
   - *Choose an importance function $g^{(1)}$;*
   - *Simulate $x_1^{(1)}, \ldots, x_n^{(1)}$ independently from $g^{(1)}$;*
   - *Compute the importance weights $w_1^{(1)}, \ldots, w_n^{(1)}$.*
2. *For $t > 1$:*
   - *Update the importance function to $g^{(t+1)}$, using the previous weighted sample $(x_1^{(t)}, w_1^{(t)})$, $\ldots$, $(x_n^{(t)}, w_n^{(t)})$.*
   - *Simulate $x_1^{(t+1)}, \ldots, x_n^{(t+1)}$ independently from $g^{(t+1)}$;*
   - *Compute the importance weights $w_1^{(t+1)}, \ldots, w_n^{(t+1)}$.*

For the present application, the most interesting result is that an approximate sample from the target density can be obtained by sampling with replacement from $g^{(t)}$ and weighting the observations by means of the vector $\boldsymbol{w}^{(t)}$.

From the theoretical point of view, it can be shown (Wraith *et al.*, 2009, Appendix A) that, on average, the algorithm monotonically decreases the cross-entropy (4). This is indeed an EM-type algorithm, and this result parallels the theorem stating that the EM algorithm for likelihood maximization increases the likelihood at each iteration. Given the strict relationship between likelihood and entropy (Asmussen *et al.*, 2005, p. 62), this result is not surprising.

The analogy with the EM algorithm can be made more precise. The Adaptive IS algorithm is based on Monte Carlo simulation, similarly to the Monte Carlo EM (MCEM) introduced by Wei and Tanner (1990). Whereas standard EM is monotone, MCEM is not. Analogously, in the present setup, for finite sample size there is no guarantee that $K(f||g^{t+1}) \leq K(f||g^t))$. However, this property is valid as $n \to \infty$; in other words, up to the variability introduced by the simulation performed at each iteration, the algorithm decreases the Cross-Entropy at each iteration. For finite n, we can also say that the algorithm is monotone on average.

In order to determine a stopping criterion, the goodness of the approximation must be evaluated at each iteration. It is convenient, and consistent with the entropy-based methodology used for developing the algorithm, to compute a measure directly related to the relative entropy (4). Thus, an approximate diagnostic is the so-called normalized perplexity $\text{perp}_n = \exp(H_n)/n$, where $H_n = -\sum_{j=1}^{n} \bar{\omega}_j \log(\bar{\omega}_j)$ is the Shannon entropy of the normalized weights $\bar{\omega}_j$. It can be shown (Cappé *et al.*, 2008, sect. 2.1) that perp_n is an estimator of $\exp(K(f||g^{(2)})$ and that is included in the interval $[0, 1]$. It seems therefore reasonable to stop the algorithm when the normalized perplexity cannot be further increased (so that the entropy cannot be further decreased). Thus, a possible stopping rule consists in monitoring the perplexity over some (say 5) successive iterations and stopping the algorithm when the normalized perplexity does not change significantly or a predefined 'large' number of iterations M is reached.

2.3. The algorithm in the Gaussian case

From the technical point of view, the fundamental issue is the update of the importance function g at each iteration. Therefore, we now detail how the

parameters of g are updated when it is a multivariate Normal mixture.

At iteration t, the importance weights associated with the sample $(\boldsymbol{x}_1^{(t)}, \ldots, \boldsymbol{x}_n^{(t)})$ are given by

$$w_j^{(t)} = \frac{f(\boldsymbol{x}_j^{(t)})}{\sum_{d=1}^{D} \pi_d^{(t)} \phi(\boldsymbol{x}_j^{(t)}; \boldsymbol{\mu}_d^{(t)}, \boldsymbol{\Sigma}_d^{(t)})}, \quad j = 1, \ldots, n,$$

and the normalized weights are $\bar{w}_j^{(t)} = w_j^{(t)} / \sum_{j=1}^{n} w_j^{(t)}$. When the instrumental density is a p-dimensional normal density, Cappé *et al.* (2004) show that the update is performed iterating, for any $i = 1, \ldots, D$, the following equations:

$$\pi_i^{(t+1)} = \sum_{j=1}^{n} \bar{w}_j^{(t)} \tau_i(\boldsymbol{x}_j^{(t)}; \pi^{(t)}, \boldsymbol{\mu}^{(t)}, \boldsymbol{\Sigma}^{(t)}), \tag{6}$$

$$\boldsymbol{\mu}_i^{(t+1)} = \frac{\sum_{j=1}^{n} \bar{w}_j^{(t)} \boldsymbol{x}_j^{(t)} \tau_i(\boldsymbol{x}_j^{(t)}; \pi^{(t)}, \boldsymbol{\mu}^{(t)}, \boldsymbol{\Sigma}^{(t)})}{\pi_i^{(t+1)}}, \tag{7}$$

$$\boldsymbol{\Sigma}_i^{(t+1)} = \frac{\sum_{j=1}^{n} \bar{w}_j^{(t)} (\boldsymbol{x}_j^{(t)} - \boldsymbol{\mu}_i^{(t+1)})(\boldsymbol{x}_n^{(t)} - \boldsymbol{\mu}_i^{(t+1)})' \tau_i(\boldsymbol{x}_j^{(t)}; \pi^{(t)}, \boldsymbol{\mu}^{(t)}, \boldsymbol{\Sigma}^{(t)})}{\pi_i^{(t+1)}}, \tag{8}$$

where

$$\tau_i(\boldsymbol{x}; \pi, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{\pi_i \phi(\boldsymbol{x}; \boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)}{\sum_{d=1}^{D} \pi_d \phi(\boldsymbol{x}; \boldsymbol{\mu}_d, \boldsymbol{\Sigma}_d)}. \tag{9}$$

As pointed out above, these equations closely resemble the EM algorithm for likelihood maximization: in particular, (6), (7) and (8) correspond to the M step, and (9) to the E step.

3. Copula families and simulation methods

3.1. Copula families

A copula is a multivariate distribution function whose univariate margins have a uniform distribution on $(0, 1)$. The importance of copulas is mostly related to Sklar's (1959) theorem. If $\boldsymbol{X}$ is a continuous p-variate random vector whose distribution function is $F(x_1, \ldots, x_p; \boldsymbol{\theta})$ with j-th margin $F_j(x_j; \theta_j)$, the theorem says that there is a unique distribution function $C : [0, 1]^p \to [0, 1]$ that satisfies

$$F(x_1, \ldots, x_p; \boldsymbol{\theta}) = C(F_1(x_1; \theta_1), \ldots, F_p(x_p; \theta_p); \boldsymbol{\alpha}). \tag{10}$$

$C(u_1, \ldots, u_p; \boldsymbol{\alpha})$ is called the copula associated to F and $\boldsymbol{\alpha}$ is the vector of its parameters.

It is worth stressing that, given $C : [0,1]^p \to [0,1]$, there is a unique distribution function F such that (10) holds. Thus, for any choice of the univariate distribution functions $F_j(x)$, even though the copula remains the same, we obtain a different multivariate distribution.

Several copulas are commonly used in practice: see Joe (1997, chap. 5). Among the most important ones, we mention here: (i) Gaussian and Student-t, known as elliptical copulas; (ii) Frank, Gumbel and Clayton, members of the family of Archimedean copulas; (iii) Galambos and Hüsler-Reiss, examples of Extreme Value copulas.

3.1.1. Extreme Value copulas

For the purposes of actuarial and financial modelling and for risk measurement, the family of Extreme Value (EV) copulas is quite important. These copulas arise naturally in the context of Multivariate Extreme Value (MEV) theory. Without entering into details here (see McNeil *et al.*, 2005, sect. 7.5, and the references therein), the building blocks are as follows.

If the normalized univariate maxima of a random vector $\boldsymbol{X}$ converge to some multivariate distribution function H, H is referred to as a MEV distribution. It can be shown that:

(i) According to univariate EV theory, the margins of H follow a Generalized Extreme Value Distribution (GEV);

(ii) From Sklar's theorem, H has a unique copula;

(iii) This kind of copula must satisfy the scaling condition $C(\boldsymbol{u}^t) = C^t(\boldsymbol{u})$ for $\boldsymbol{u} \in [0,1]^p$ and for any $t > 0$ (McNeil *et al.*, 2005, Theorem 7.44).

Among the most common copulas, those that satisfy (iii) are the Gumbel, Galambos, Hüsler-Reiss and t-EV copulas, which are the most important EV copulas.

Given a multivariate distribution F, we would like to know which EV copula 'corresponds' to F. Similarly to univariate EV theory, any multivariate distribution F is in the domain of attraction of some EV copula C_{EV}, so that there is a correct EV copula to be used in practical applications, according to the dependence structure of F, as measured by the copula C of F. However, determining C_{EV} requires the knowledge of C, which is typically unknown. Fortunately, all

EV copulas are very similar to each other, so that in practice any EV copula with GEV margins approximately gives the same results.

3.2. Simulating copulas

Stochastic simulation of multivariate probability distributions is, in general, a difficult task. One reason is that the the univariate inversion method cannot be extended to the multivariate case. Moreover, the performance of other standard techniques deteriorates quickly as the dimension of the problem increases (Devroye, 1986, p. 554).

As for copulas, one can in principle use the following result (Joe, 1997, p. 146), reported here for simplicity for the bivariate case. Let C be a bivariate copula, and define $C_{2|1}$ to be the conditional distribution function $C_{2|1}(u_2|u_1)$. If U_1 and Q are independent $U(0,1)$ random variables, $(U_1, U_2) = (U_1, C_{2|1}^{-1}(Q|U_1))$ has distribution C. This technique, sometimes known as *conditional distribution approach* or *iterative conditioning*, is appealing because only univariate simulations are required. Unfortunately, for most copulas the function $C_{2|1}^{-1}$ does not exist in closed form. In this case, after sampling u_1, to obtain u_2 one has to use a root-finding routine to solve for u_2 the equation $q = C_{2|1}(u_2|u_1)$. This may not work even in the bivariate case and becomes essentially intractable when the dimension of the problem is larger.

The common way of simulating copulas is thus based on specific techniques for various classes of copulas. Broadly speaking, one can consider three different families of copulas.

- Elliptical copulas (Gaussian and Student-t copulas). For this family, simulation is easy, because it basically amounts to simulating the Gaussian and Student-t multivariate densities. For example, observations from the p-variate Gaussian copula with marginals F_i can be generated as follows:

  (1a) Simulate a random vector $\mathbf{x} = (x_1, \ldots, x_p)'$ from the $N_p(\mathbf{0}, \boldsymbol{R})$ distribution;

  (1b) Evaluate $u_k = \Phi(x_k)$, for any $k = 1, \ldots, p$;

  (2) Compute $y_k = F_k^{-1}(u_k; \boldsymbol{\theta}_k)$ for any $k = 1, \ldots, p$;

  (3) Repeat steps (1)-(2) a large number of times B.

- LT-Archimedean copulas (McNeil *et al.* , 2005, sect. 5.4.2). They have the property that the Laplace Transform (LT) of the inverse of the generator

exists in closed form. A general procedure for simulating such copulas is based on a technique given in McNeil *et al.* (2005, p. 223); see also Yan (2007, sect. 3.2). The approach is based on some results first obtained by Marshall and Olkin (1988) and requires generating random numbers from some positive random variable K, often called frailty: in particular, for the simulation of Clayton, Frank, and Gumbel copulas K is respectively gamma, log-series, and positive stable. Whereas the first case is straightforward, the last two cases are not completely trivial. Moreover, the fact that K is different for each copula makes this way of proceeding somewhat impractical.

- For the remaining copulas, essentially no method is available except the conditional distribution approach, which is complicated because $C_{2|1}^{-1}$ is usually unknown.

4. Simulation of copulas by means of Adaptive IS

The implementation of Adaptive IS for the simulation of copulas requires to choose the class of the importance function and derive the so-called updating equations, i.e. the equations resulting from the minimization of the cross-entropy (4). For the Gaussian case and a generic target f, the details of the algorithm have been given in sect. 2.3. When turning to copulas, f is the density of the copula and the Adaptive IS algorithm based on Gaussian mixtures works as follows.

1. Choose the instrumental density $g^{(1)} = \sum_{d=1}^{D} \pi_d \phi(\boldsymbol{\mu}_d, \boldsymbol{\Sigma}_d)$, i.e. the parameters D, $\boldsymbol{\pi}$, $\boldsymbol{\mu}_i$, $\boldsymbol{\Sigma}_i$ $(i = 1, \ldots, D)$.
2. Simulate $\boldsymbol{x}_1^{(1)}, \ldots, \boldsymbol{x}_n^{(1)}$ independently from $g^{(1)}$.
3. Compute the importance weights $w_j^{(1)} = f(x_{1j}^{(1)}, \ldots, x_{pj}^{(1)}) / g(x_{1j}^{(1)}, \ldots, x_{pj}^{(1)})$ $(j = 1, \ldots, n)$. Notice that f is the density function $c(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n; \boldsymbol{\theta}, \boldsymbol{\alpha}) = c(F_1^{-1}(\boldsymbol{x}_1; \theta_1), \ldots, F_p^{-1}(\boldsymbol{x}_p; \theta_p); \boldsymbol{\alpha}) f_1(\boldsymbol{x}_1; \theta_1), \ldots, f_p(\boldsymbol{x}_p; \theta_p)$, namely the target density expressed in terms of the copula density c, the marginal distribution functions F_k and the marginal densities f_k $(k = 1, \ldots, p)$.

For $t \geq 1$:

4. Update the importance function to $g^{(t+1)}$ using equations (6) to (9) and the previous weighted sample $(\boldsymbol{x}_1^{(t)}, w_1^{(t)})$, …, $(\boldsymbol{x}_N^{(t)}, w_N^{(t)})$.

5. Simulate $\boldsymbol{x}_1^{(t+1)}, \ldots, \boldsymbol{x}_N^{(t+1)}$ independently from $g^{(t+1)}$;
6. Compute the importance weights $w_j^{(1)}$ $(j = 1, \ldots, n)$;
7. Go back to step 4.

In the Student-t case the preceding steps remain unchanged, but the Gaussian density must be replaced by the Student-t density and equations (6) to (9) in step 4. are different. We do not write explicitly the analogues of (6) to (9) for a Student-t mixture (the interested reader is referred to Cappé *et al.*, 2008, sect. 4). However, it is worth adding that, in the Student-t setup, the number of degrees of freedom cannot be estimated, so that the IS density is not completely adaptable.

The choice of the mixture parameters D, $\boldsymbol{\pi}$, $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ $(i = 1, \ldots, D)$, of the sample size n, and of the maximum number of iterations M depends on several factors. Some rules of thumbs are valid in any setup, but more precise evaluations cannot abstract from specific properties of the target density and are therefore delayed to the next section.

As concerns D, a larger value produces a more precise approximation. Roughly speaking, in general it can be said that the farther away from ellipticity is the target density, the larger is the number of components necessary to achieve a good approximation. Cappé *et al.* (2008) mostly employ 3 or 4 components. A useful strategy consists in starting with a 'large' value of D an discard components with a too small weight, such as < 0.002. The value of D is also related to the choice of n. If both numbers are small, adaptation may take a long time and/or fail to produce a reasonable approximation of the target density. It follows that the maximum number of iterations should also be set according to n and D: typically, the perplexity reaches relatively large values (how large depends on the target density) in few iterations, but further improvements take many iterations, so that, for a precise approximation, a much larger M is needed. In our experience, even a very small improvement of a relatively large perplexity may be important, in particular when the target density is heavy-tailed; in such a case the bulk of the density is quickly well matched (therefore producing a large perplexity) but, if the number of iterations is too small, the tails may be underestimated.

Wraith *et al.* (2009) use, for their application, $5 \leq D \leq 10$ and $7500 \leq n \leq 10\,000$. Concerning the problem at hand, the experiments in the next section are devoted to the assessment of the impact of the numerical values of these

parameters for various copula-based distributions.

As for $\boldsymbol{\pi}$, $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$, a good choice of the initial instrumental density speeds up the convergence of the algorithm. For adaptation to be effective, the instrumental density should visit all the regions of the target density; thus, it is particularly important that the tails of the target are explored by the instrumental density. The main implication is that one has to use the 'right' family of instrumental densities: for example, for an heavy-tailed target, a Student-t mixture with a small number of degrees of freedom should be preferable. Having chosen an instrumental density that, for some parameter configuration, provides a close approximation to the target, the next step consists in selecting the initial values of the parameters. When no or little information is available about the target, a starting guess may be obtained by means of the maximum likelihood estimators of the parameters. If, as in the present application, the shape of the target density is known, it is also possible to choose the parameters so that the shape of the instrumental density resembles the target. Similarly to the standard EM algorithm, a bad initialization may considerably slow down the algorithm, but typically does not prevent it from converging.

4.1. Simulation of Extreme Value Copulas

The simulation of extreme value copulas is less straightforward and requires an appropriate transformation of the marginals. To formalize the procedure, some preliminary review of the GEV distribution (see Kotz and Nadarajah, 2000, for details) is needed. The three parameters of the GEV distribution are usually denoted by μ (location), σ (scale) and ξ (shape); its density function is given by

$$f_{GEV}(x) = \frac{1}{\sigma} t(x)^{\xi+1} \exp(-t(x)),$$

where

$$t(x) = \begin{cases} (1 + \xi(x-\mu)/\sigma)^{-1/\xi} & \xi \neq 0; \\ \exp(-(x-\mu)/\sigma) & \xi = 0. \end{cases}$$

The support is $[\mu - \sigma/\xi, +\infty)$ when $\xi > 0$, $(-\infty, +\infty)$ when $\xi = 0$ and $(-\infty, \mu - \sigma/\xi]$ when $\xi < 0$. The distribution is generalized because it corresponds to the Gumbel distribution (or type 1 GEV) when $\xi = 0$, to the Fréchet (or type 2 GEV) when $\xi > 0$, and to the Weibull (or type 3 GEV) when $\xi < 0$,

When trying to use Adaptive IS for simulating an EV copula, there is a difficulty because its support is not $\mathbb{R}^p$ but the p-dimensional interval $[\mu_1 -$

$\sigma_1/\xi_1, +\infty) \times \cdots \times [\mu_p - \sigma_p/\xi_p, +\infty)$. On the other hand, the support of a p-dimensional Gaussian or Student-t mixture is $\mathbb{R}^p$. Although Adaptive IS only requires the support of the instrumental density to include the support of the target density, it is clear that, at each iteration, all the observations sampled from the mixture with at least one component smaller than the corresponding lower bound are wasted (their weight is equal to zero) and this makes the method quite inefficient.

There is, however, a way around this problem. It is based on the following two results.

Proposition 1. *Let* $X \sim \mathrm{Fr}(\mu, \sigma, \xi)$ *and* $Y = 1 + \xi(X - \mu)/\sigma$. *Then* $\log(Y) \sim \mathrm{Gumb}(0, \xi)$.

Proof. Immediate using basic tools for transformations of random variables.

The random variable Y introduced in Proposition 1 is a standard 'Translated Fréchet' (TFr) with location 0, scale 1, shape ξ and support $[0, +\infty)$ (Embrechts *et al.*, 1997, sect. 3.4).

Proposition 2. *Let* $(X_1, \ldots, X_p)'$ *be a random vector with continuous margins and copula* C, *and let* $T_1, \ldots, T_p$ *be strictly increasing functions. Then* $(T_1(X_1), \ldots, T_p(X_p))'$ *also has copula* C.

Proof. See McNeil *et al.* (2005), Proposition 5.6.

Using propositions 1 and 2 we obtain Algorithm 3 for sampling EV copulas.

Algorithm 3. *Consider simulating an EV copula with density* c_{EV}, *parameter* δ *and Fréchet margins* X_{1i} *with parameters* μ_i, σ_i *and* $\xi_i > 0$ *(*$i = 1, \ldots, p$*). One can proceed as follows.*

1. *Let* $X_{2i} = 1 + \xi_i(X_{1i} - \mu_i)/\sigma_i$. *Then* $X_{2i} \sim \mathrm{TFr}(0, 1, \xi_i)$.
2. *Let* $Y_i = \log(X_{2i})$. *According to Proposition 1,* $Y_i \sim \mathrm{Gumb}(0, \xi_i)$.
3. *Use Adaptive IS to simulate the EV copula* c_{EV} *with Gumbel margins* $Y_1, \ldots, Y_p$.
4. *Let* $V_{1i} = \exp(Y_i)$. *Using again Proposition 1,* $V_{1i} \sim \mathrm{TFr}(0, 1, \xi_i)$.
5. *Let* $V_{2i} = \mu_i + \sigma_i/\xi_1(V_{1i} - 1)$. *Clearly,* $V_{2i} \sim \mathrm{Fr}(\mu_i, \sigma_i, \xi_i)$. *Proposition 2 implies that* V_{2i} *(*$i = 1, \ldots, p$*) are observations from the EV copula* c_{EV} *with Fréchet marginals.*

5. Simulation and application

5.1. Simulation

In this section we perform some simulations to assess the performance of the algorithm and evaluate the impact on the output of the choices of the initial parameters.

In the experiments we always use a Gaussian mixture as IS density. Student-t mixtures were tested but always gave worse results. In particular, for all copulas, the Student-t mixture gives overdispersed simulated marginal distributions, namely it is 'too leptokurtic'. Moreover, the excess kurtosis of the simulated data seems to decrease monotonically and the perplexity to increase monotonically as the number of degrees of freedom tends to infinity, that is, as the population densities tend to the Normal.

For all copulas, we put $n = 1000$; the algorithm stops after a number of iterations equal to $\min\{20, t^*\}$, where $t^* > 5$ is such that $\max_{i,j \in \{t^*, t^*-1, \ldots, t^*-5\}} |\text{perp}_i - \text{perp}_j| < \epsilon$, with $\epsilon = 0.001$. The results are quite good with D equal to four or five; however, the increase in computational burden caused by the addition of more components is negligible, so that we set $D = 7$. The instrumental density actually used for the simulation is the mixture corresponding to the largest perplexity. We repeat the simulation $B = 500$ times. We tried several starting values and, as expected, their role turned out to be essentially irrelevant: in case of bad initializations, just the first two or three iterations are affected, in the sense that the perplexity is low. In the next iterations, the perplexity quickly increases, reaching the same levels of the cases when initialization is better (see also Fig. 2). For all copulas, we employ the following starting values:

$$\boldsymbol{\pi} = \begin{pmatrix} 1/7 \\ 1/7 \\ 1/7 \\ 1/7 \\ 1/7 \\ 1/7 \\ 1/7 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} -0.5 & -0.5 \\ -0.3 & -0.3 \\ -0.2 & -0.2 \\ 0 & 0 \\ 0.4 & 0.4 \\ 0.5 & 0.5 \\ 0.6 & 0.6 \end{pmatrix}, \quad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_3 = \boldsymbol{I}_2,$$

$$\boldsymbol{\Sigma}_4 = \boldsymbol{\Sigma}_5 = \boldsymbol{\Sigma}_6 = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix} \quad \boldsymbol{\Sigma}_7 = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}.$$

The experiments simulate the Gumbel copula with $\delta = 2$, the Frank copula with $\alpha = 5.7$, the Galambos copula with $\delta = 1.5$ and the Hüsler-Reiss copula with $\delta = 1.5$. In the first two cases, the marginals are standard Normal, whereas in the last two cases the marginals are $\text{Fr}(0, 1, 0.5)$.

Fig. 1(a) and 1(b) respectively show the contour plot of the Gumbel and Frank copula density and of the instrumental density. The latter is the best

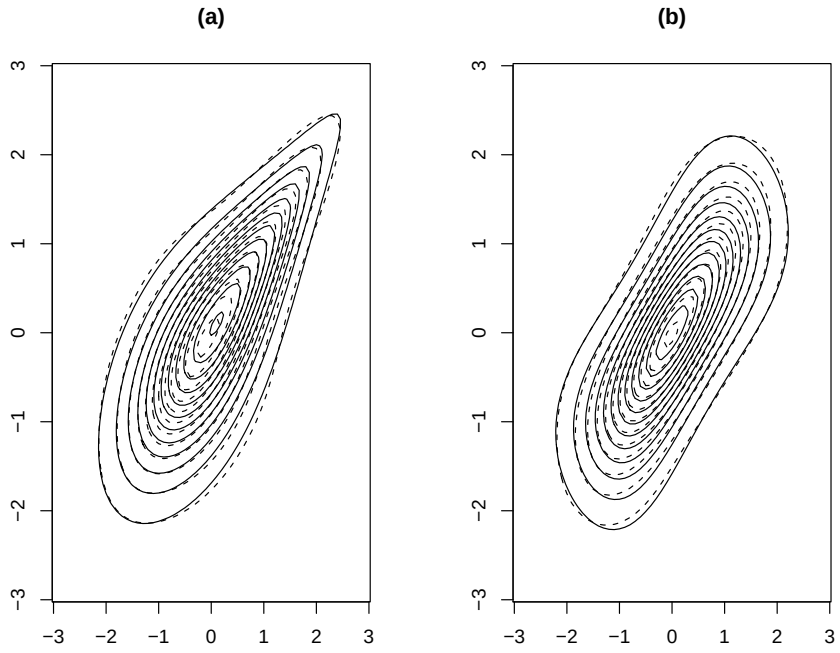

Figure 1: Contours of Gumbel (a) and Frank (b) copulas with standard Normal margins and best fitting Gaussian mixture.

bivariate normal mixture, obtained by means of the Adaptive IS procedure.

Some sample measures concerning the simulated distributions are reported in Table 1. The figures in the table are the MC estimates of the expected value, the variance, the skewness and the kurtosis of the marginal distributions, as well as of the copula parameter; 'max perp' is the average of the largest perplexities obtained at each replication. Similarly to Joe (1997, p. 146), at each replication of the experiment the value of the copula parameter is estimated by first computing numerically the Kendall's correlation coefficient τ using the simulated data, and then finding the corresponding value of δ. Both the marginal distributions (for which an assessment of precision can be based on the comparison of sample mean, variance, skewness and kurtosis and of their theoretical counterparts) and the dependence structure (measured by the coefficient of the copula)

are very well approximated. Note that, in the Galambos and Hüsler-Reiss cases, we compute the sample moments of the Y_is (step 3 of Algorithm 3), which are supposed to follow a Gumbel distribution, and compare them to their theoretical counterparts, reported in the sixth and eighth column of the table. This is of course equivalent to analysing the V_is (step 5 of Algorithm 3), because they are obtained from the Y_is by means of a deterministic transformation that does not alter the statistical properties of the simulated data.

	Gumbel		Frank		Galambos		HR	
	True	Sim.	True	Sim.	True	Sim.	True	Sim.
$\mathrm{E}(X_1)$	0	-0.004	0	-0.001	0.289	0.285	0.289	0.287
$\mathrm{E}(X_2)$	0	-0.003	0	0.000	0.289	0.287	0.289	0.290
$\mathrm{var}(X_1)$	1	1.003	1	1.006	0.411	0.406	0.411	0.411
$\mathrm{var}(X_2)$	1	1.005	1	1.002	0.411	0.412	0.411	0.421
sk_1	0	0.003	0	0.001	≈ 1.14	1.101	≈ 1.14	1.123
sk_2	0	-0.002	0	0.007	≈ 1.14	1.112	≈ 1.14	1.158
k_1	3	3.008	3	3.013	5.4	5.372	5.4	5.305
k_2	3	3.019	3	3.029	5.4	5.318	5.4	5.344
cop. par.	2	2.004	5.7	5.673	1.5	1.515	1.5	1.494
max perp		0.997		0.996	1.5	0.995	1.5	0.995

Table 1. Results of the simulation experiments. The numbers in the columns labelled 'True' are the expected value, variance, skewness and kurtosis of the marginal distributions, namely standard Normal for the Gumbel and Frank copulas and Gumbel$(0, 0.5)$ for the Galambos and Hüsler-Reiss (HR) copulas.

Further experiments with different numerical values of the parameters (in particular, different σ when the marginals are Gaussian and different ξ when the marginals are GEV) did not show any difference with respect to the standard cases considered above and are therefore not reported here.

Figure 2 displays the boxplots of the perplexities over the first 6 iterations of the algorithm for the Gumbel and the Galambos cases: it can be seen that adaptation is very quick.

Before turning to real data, we report the results of a further experiment aimed at assessing the speed of convergence. For the Galambos copula with GEV$(0, 1, 0.5)$ marginals, we repeated $B = 50$ times the simulation for each $n = (500, 1000, 3000, 5000, 7500, 10000)$. Fig. 3 displays the estimates of the expected value (a) and the variance (b) of the first Gumbel marginal, of the copula parameter δ (c), and of the perplexity (d). The approximation is accept-

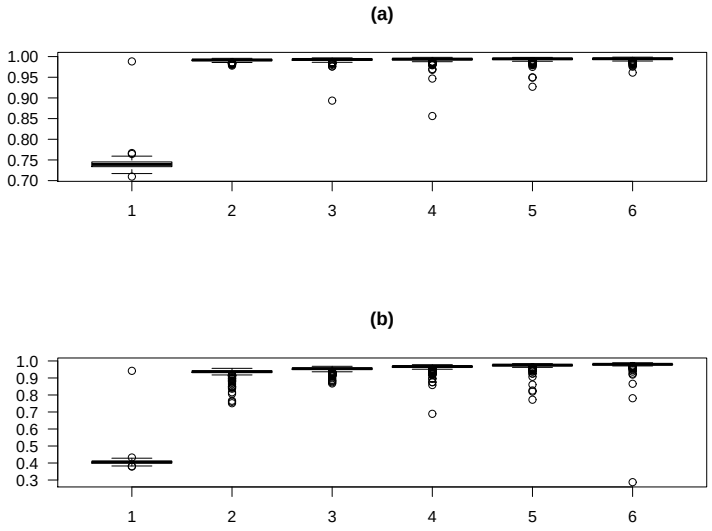

Figure 2: Normalized perplexity estimates in the first 6 iterations of the algorithm for the Gumbel copula with standard Normal margins (a) and the Galambos copula with Gumbel $(0, 0.5)$ marginals (b).

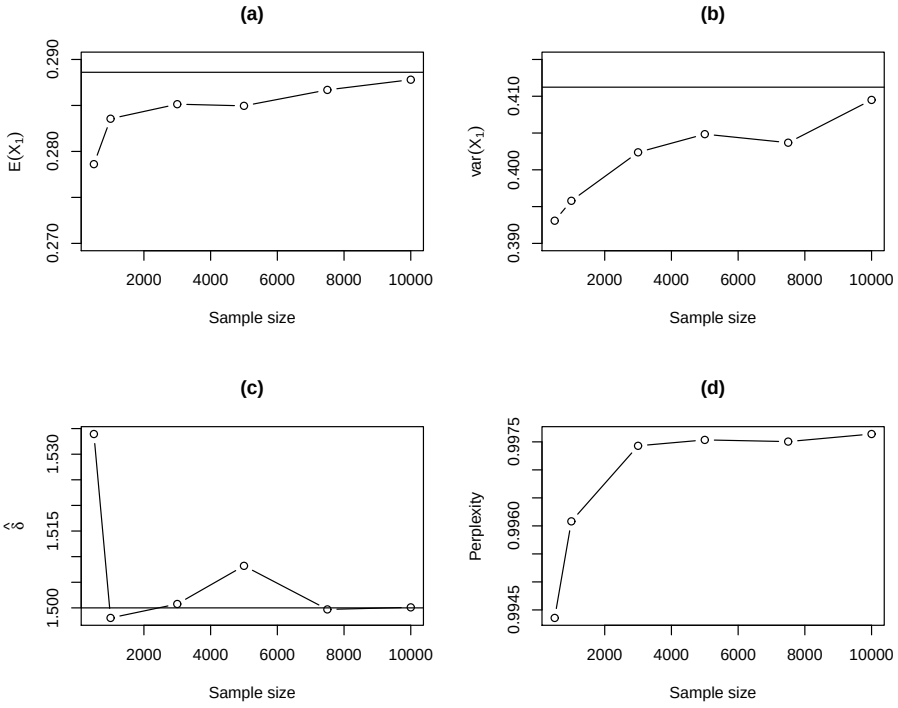

Figure 3: Estimates of the expected value (a) and the variance (b) of the first Gumbel marginal, of the copula parameter δ (c), and of the perplexity (d) as a function of the sample size. The horizontal line represents the reference value.

able with $n = 500$ and quite satisfactory for $n \geq 1000$. With $n = 10\,000$ the level of precision is very high.

5.2. Application

In this section we perform an empirical analysis concerning the joint distribution of the maxima of the time series of the closing prices S_1 and S_2 of the Microsoft and Sun Microsystems stocks. The time horizon is January 1, 2001, to December 31, 2009. We are ultimately interested in the joint tail probability of S_1 and S_2, defined as $p_{12} = P(S_1 > c_1, S_2 > c_2)$ for some 'large' values c_1 and c_2.

The first step consists in estimating the parameters of the two GEV univariate distributions. According to the Block Maxima approach, we compute the

monthly maxima of each series, obtaining a sample of 108 observations, ranging respectively in $[19.31, 73.68]$ and $[3.09, 33.56]$. Maximum likelihood estimates of the parameters are shown in Table 2; standard errors are based on a numerical approximation of the observed information matrix (see Smith, 1985, for details).

	S_1	S_2
$\hat{\mu}$	27.77 (0.70)	5.02 (0.19)
$\hat{\sigma}$	6.58 (0.64)	1.78 (0.21)
$\hat{\xi}$	0.42 (0.08)	0.67 (0.10)

Table 2. Estimates of the parameters of the marginal GEV distributions.

Thus, both marginal distributions are Fréchet. As anticipated above, the functional forms of common EV copulas are very similar to each other, so that the choice of the EV copula to be used in practice is not relevant. We tried to implement both the Gumbel and the Galambos copula, and the difference was negligible. Here we show the results obtained with the Galambos copula.

The Maximum Pseudo-Likelihood estimate of the copula parameter δ is $\hat{\delta} = 0.605$ with a standard error equal to 0.122. Considering that the Galambos copula has upper tail dependence[1] (McNeil *et al.*, 2005, sect. 7.5.1), an estimate $\hat{\delta}$ significantly different from zero empirically confirms that the joint distribution of S_1 and S_2 has tail dependence. Given that the two stocks belong to the same sector, it is not surprising that their extreme values are dependent. Being able to model this type of dependence is a considerable added value with respect to, for example, the multivariate Normal distribution, which does not have upper tail dependence.

Following the procedure outlined in Section 4.1 with $n = 10\,000$, $M = 80$ and $\epsilon = 10^{-5}$ we get the results in Fig. 4 and Table 3. Fig. 4(a) shows the contour plot of the estimated distribution and of the approximating mixture. Fig. 4(b) displays a simulated sample of size $n = 1000$. Fig. 4(c) and (d) display the simulated marginal distributions and the theoretical Gumbel densities. Finally, Table 3 gives the values of the means, the variances and the copula parameter δ estimated from the real data ('True') and from the simulated data ('Simulated'). From Fig. 4 and Table 3 the quality of the simulated data is very good.

[1]Roughly speaking, this means that positive large values are 'strongly dependent'; see McNeil *et al.*, 2005, sect. 5.2.3, for a formal definition.

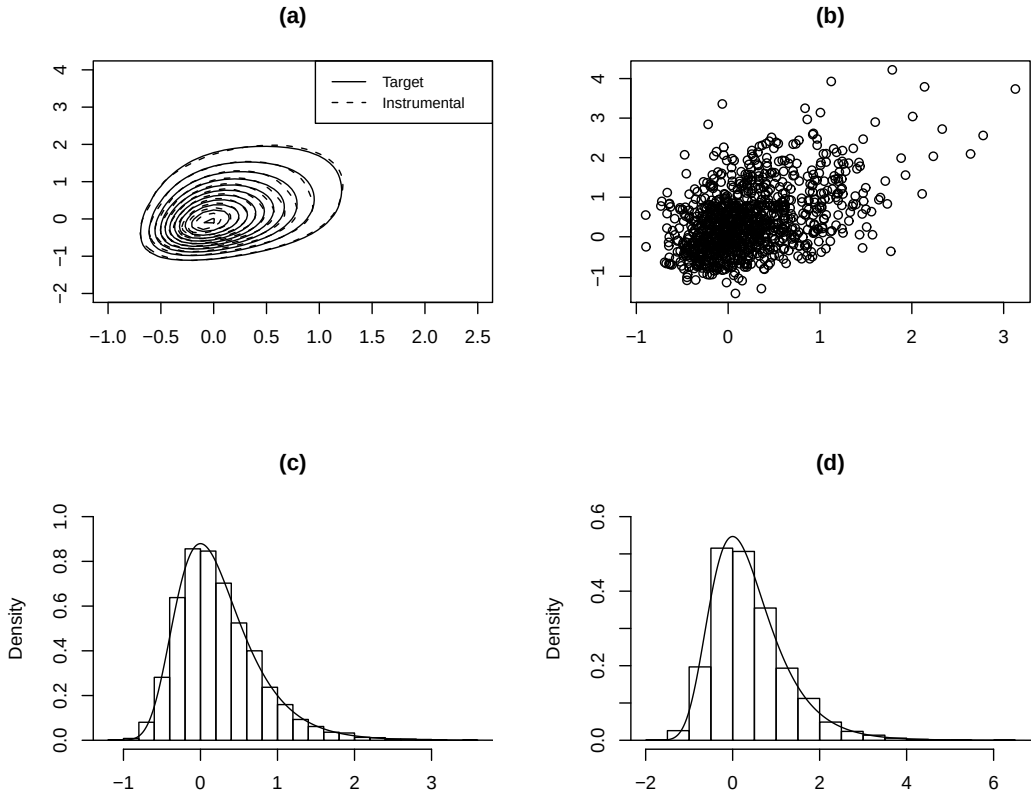

Figure 4: (a) Contours of the estimated Galambos copula and best fitting Gaussian mixture; (b) a simulated sample of size $n = 1000$; (c) simulated and estimated marginal distribution of S_1; (d) simulated and estimated marginal distribution of S_2.

	$\mathrm{E}(S_1)$	$\mathrm{var}(S_1)$	$\mathrm{E}(S_2)$	$\mathrm{var}(S_2)$	δ
True	0.241	0.288	0.388	0.745	0.605
Simulated	0.238	0.287	0.387	0.748	0.619

Table 3. Estimates of the expected values, the variances and the copula parameter obtained from the observed data (first line) and from the simulated data (second line).

Finally, we would like to estimate the joint probability p_{12} for large c_1 and c_2. As pointed out by McNeil *et al.* (2005, Example 7.52), in an analytical approach based on the marginals and/or the joint distribution functions, it would only be possible to compute the probabilities $p_1 = P(S_1 > c_1)$, $p_2 = P(S_2 > c_2)$ and $p_{1\cup 2} = P(S_1 > c_1 \cup S_2 > c_2)$. In a simulation approach, this difficulty is overcome. Table 4 shows the results obtained for various thresholds.

(c_1, c_2)	$(73, 33)$	$(73, 38)$	$(80, 33)$	$(80, 38)$	$(85, 38)$	$(90, 40)$	$(100, 50)$
p_{12}	0.0093	0.0080	0.0086	0.0073	0.0065	0.0057	0.0043

Table 4. Estimates of the joint tail probability $P(S_1 > c_1, S_2 > c_2)$ for various values of c_1 and c_2.

It can be seen that p_{12} remains positive for all the values of the thresholds considered. The possibility of estimating tail probabilities well beyond the maximum of the data is one of the most important advantages of univariate EVT, preserved also in multivariate EVT.

6. Conclusions

This paper uses a generalization of Importance Sampling, called Adaptive Importance Sampling, for sampling copula-based probability distributions and computing tail risk measures. The algorithm works for any copula-based distribution, thus overcoming the difficulties encountered in simulating non-elliptical and non-Archimedean copulas. We first simulate two classical Archimedean copulas (Gumbel and Frank) with Normal margins and then, using the link between the Gumbel and the Fréchet distribution, two EV copulas, namely the Galambos and Hüsler-Reiss copulas with Fréchet marginals. The simulation analysis and a real-data application confirm the excellent performance of the procedure.

Adaptive IS allows to solve one of the main problems of classical IS, namely the use of a single, fixed instrumental density. Moreover, it is intuitively appealing because it exploits the ability of mixtures of distributions at approximating other distributions. Given the strict relationship between likelihood maximization and cross-entropy minimization, the updating equations turn out to be very similar to those of the EM algorithm.

In the application presented in this paper, the strength of this approach consists mainly in its generality. This is a considerable advantage over existing simulation procedures, which use, when available, different algorithms. Moreover, the experiments show that the sensitivity to the starting values of the parameters is low, and even a bad initialization does not prevent the algorithm from converging. Finally, monitoring the 'convergence' of the algorithm is simple, as the typically used diagnostic, called normalized perplexity, is easy both to compute and to interpret.

The algorithm can be used for the simulation of other non-standard multivariate distributions for which the density function is known in closed form, and further research is underway to apply the algorithm to the simulation of the Generalized Hyperbolic Distribution.

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